Stochastic Programming in GAMS

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GAMS at a Glance

General Algebraic Modeling System

- Algebraic Modeling Language
- 30+ Integrated Solvers
- 10+ Supported MP classes
- 10+ Supported Platforms
- Connectivity- & Productivity Tools
  - IDE
  - Model Libraries
  - GDX, Interfaces & Tools
  - Grid Computing
  - Benchmarking
  - Compression & Encryption
  - Deployment System
  - APIs (C, Fortran, Java, .Net ...)
  - ...
2010
Example Model: Gas Price Model

Gas Storage

Inject/
Buy

Withdraw/
Sell
n-Stage Stochastic Programs

• Construct Scenario Tree:
  – Start with today’s price and use a (discrete) distribution
  – Realizations: up, down
• Stochastic Linear Program (block structure)
  – Nested Bender’s Decomposition (OSLSE, FortSP, AIMMS)
  – In practice Deterministic Equivalent with Barrier method

\[
Z_{RN} = \min_{x_i} \left\{ c_1x_1 + E_{x_2} \left[ \min_{x_2} c_2x_2 + E_{x_3} \left[ \min_{x_3} c_3x_3 + \ldots + E_{x_T} \left[ \min_{x_T} c_Tx_T \right] \right] \right] \right\}
\]

subject to:

\[
\begin{align*}
A_{i1}x_1 + A_{i2}x_2 + \ldots + A_{ih}x_h &= b_i \\
\ell_i &\leq x_i \leq u_i
\end{align*}
\]
ScenRed (Römisch et. al., HU Berlin)

- Find good approximation of original scenario tree of significant smaller size
- Available since 2002
- Integrated in GAMS system
- No extra cost
Tree Generation: ScenRed2

• Construct a true scenario tree from independent scenarios:

• Reconstruct underlying distribution from a set of scenarios
2-Stage Stochastic Programs

- SP Solver DECIS (Gerd Infanger, Stanford, USA)
  - Stores only one instance of the problem and generates scenario sub-problems as needed
  - Solution Strategies
    - Deterministic Equivalent (all scenarios)
    - Sampling:
      - Crude Monte Carlo/
      - Importance sampling
AML and Stochastic Programming (SP)

• Algebraic Modeling Languages/Systems good way to represent optimization problems
  – Algebra is a universal language
  – Hassle free use of optimization solvers
  – Simple connection to data sources (DB, Spreadsheets, …) and analytic engines (GIS, Charting, …)

• Large number of (deterministic) models in production
  – Opportunity for seamless introduction of new technology like Global Optimization, Stochastic Programming, …
  – AML potential framework for SP
Simple Example
A Transportation Model

Seattle (350)

San Diego (600)

Chicago (300)

New York (325)

Topeka (275)
Simple Example: Transportation Model

Data:
- Certain capacity at plants
  \[ a(i) / \text{seattle} 350, \text{san-diego} 600 / \]
- Certain demand at markets
  \[ b(j) / \text{new-york} 325, \text{chicago} 300, \text{topeka} 275 / \]
- Given transportation cost
  \[ \begin{array}{ccc}
  \text{new-york} & \text{chicago} & \text{topeka} \\
  \text{seattle} & 0.225 & 0.153 & 0.162 \\
  \text{san-diego} & 0.225 & 0.162 & 0.126 \\
  \end{array} \]
- Units can also be bought at markets directly for a fixed price
  \[ p / 1 / \]

Decisions:
- How many units to ship:
  \[ X(i,j) \]
- How many units to buy:
  \[ U(j) \]
... in order to minimize total cost
Transportation Model – GAMS Formulation

* Costs to minimize

\[ \text{cost..} \quad Z = e= \ \text{sum}( (i,j), \ c(i,j) \ast X(i,j)) \]
\[ + \ \text{sum}( \ j, \ p \ast U(j)) ; \]

* Supply limitation

\[ \text{supply(i)..} \ \text{sum}(j, \ X(i,j)) = l= a(i); \]

* Demand requirement

\[ \text{demand(j)..} \ \text{sum}(i, \ X(i,j)) = g= b(j) - U(j); \]

Model transport / all /;

Solve transport using lp minimizing Z;
Transportation Problem – Add Uncertainty

- Uncertain demand factor $bf$

- Decisions to make:
  - How many units should he shipped “here and now” (without knowing the outcome of the uncertain demand)?
    \[\Rightarrow \text{First-stage decision}\]
  - How many units need to be bought after the outcome becomes known?
    \[\Rightarrow \text{Second-stage or recourse decision}\]
  - Recourse decisions can be seen as
    - penalties for bad first-stage decisions
    - variables to keep the problem feasible
* Add demand factor \( bf \)

\[
demand(j) \quad \text{sum}(i, \, x(i,j)) = g = \bfbf \times b(j) - U(j);
\]

...

* Make \( bf \) uncertain

\[
\text{randvar } bf \text{ discrete } 0.3 \, 0.95 \\
\quad 0.5 \, 1.00 \\
\quad 0.2 \, 1.05
\]

* Define non-default stages

\[
\text{stage 2 } bf \, u \, \text{demand}
\]
New GAMS (EMP) Keywords
Excursus: EMP, what?

With new modeling and solution concepts do not:
• overload existing GAMS notation right away!
• attempt to build new solvers right away!

But:
• Use existing language features to specify additional model features, structure, and semantics
• Express extended model in symbolic (source) form and apply existing modeling/solution technology
• Package new tools with the production system

→ Extended Mathematical Programming (EMP)
JAMS: a GAMS EMP Solver

Emp Information

Original Model

Translation

Reformulated Model

Viewable

Mapping Solution into original space

Solving using established Algorithms

Solution
Random Variables

Discrete Distribution

Normal Distribution

Poisson Distribution

Exponential Distribution
Random Variables (RV) [randVar]

• Defines both discrete and parametric random variables:

  randVar rv discrete prob val {prob val}

• The distribution of discrete random variables is defined by pairs of the probability prob of an outcome and the corresponding realization val

  randVar rv distr par {par}

• The name of the parametric distribution is defined by distr, par defines a parameter of the distribution
Joint RVs \([jRandVar]\)

- Defines discrete random variables and their joint distribution:

  \[ jRandVar \text{ rv rv \{rv\} prob val val \{val\}} \]

  \[ \{prob val val \{val\}\} \]

- At least two discrete random variables \(rv\) are defined and the outcome of those is coupled
- The probability of the outcomes is defined by \(prob\) and the corresponding realization for each random variable by \(val\)
Correlation between RVs [correlation]

- Defines a correlation between a pair of random variables:

\[
\text{correlation } \text{rv} \ \text{rv} \ \text{val}
\]

- \text{rv} is a random variable which needs to be specified using the \text{randvar} keyword and \text{val} defines the desired correlation \((-1 \leq \text{val} \leq 1)\)
Stages

Stage 1: Decision X

Observation of bf

Stage 2: Decision U

...
Stages [stage]

• Defines the stage of random variables (rv), equations (equ) and variables (var):

  stage stageNo rv | equ | var {rv | equ | var}

• StageNo defines the stage number
• The default StageNo for the objective variable and objective equation is the highest stage mentioned
• The default StageNo for all the other random variables, equations and variables not mentioned is 1
Chance Constraints

OBJ.. Z =e= X1 + X2;
E1.. rv1*X1 + 2*X2 =g= 5;
E2.. rv2*X1 + 6*X2 =g= 10;
Model sc / all /;
solve sc min z use lp;

rv1

Prob: 0.5
val: 2

Prob: 0.5
val: 3

rv2

Prob: 0.33
val: 2

Prob: 0.33
val: 4

Prob: 0.33
val: 6

chance E1 0.5
chance E2 0.5
Chance Constraints

1 out of 2 must be true
[0.5 ≥ 0.5]

\[ 2 \times X_1 + 2 \times X_2 \geq 5; \]
\[ 3 \times X_1 + 2 \times X_2 \geq 5; \]

2 out of 3 must be true
[0.66 ≥ 0.5]

\[ 2 \times X_1 + 6 \times X_2 \geq 10; \]
\[ 4 \times X_1 + 6 \times X_2 \geq 10; \]
\[ 6 \times X_1 + 6 \times X_2 \geq 10; \]

Just in case: \( X_1 = 1 \) and \( X_2 = 1 \) are optimal.
Chance Constraints [chance]

• Defines individual or joint chance constraints (CC):

\[
\text{chance equ } \{\text{equ}\} \ [\text{holds}] \ \text{minRatio} \ [\text{weight|varName}]\]

• Individual CC: A single constraint \text{equ} has to hold for a certain ratio \(0 \leq \text{minRatio} \leq 1\) of the possible outcomes
• Joint CC: A set of constraints \text{equ} has to hold for a certain ratio \(0 \leq \text{minRatio} \leq 1\) of the possible outcomes
• If \text{weight} is defined, the violation of a CC gets penalized in the objective (weight violationRatio)
• If \text{varName} is defined the violation get multiplied by this existing variable
Expected Value \[\texttt{ExpectedValue}\]

- This is the default objective:

\[
\texttt{ExpectedValue} \ [x \ \texttt{EV}\_x]
\]

- If only \texttt{ExpectedValue} is defined, the expect value of the GAMS objective variable will be optimized (same as if it would be omitted at all)

- If the variable pair \(x \ \texttt{EV}\_x\) is defined, GAMS will replace its objective variable by \texttt{EV}\_x, which will become the expected value of \(x\)
Conditional Value at Risk [cVaR]

• As an alternative to the expected value, the conditional value at risk (cVaR) can be optimized:

\[ \text{cVaR} \ [x \ cVaR_x] \ \text{theta} \]

• If only \( \text{cVaR theta} \) is defined, the cVaR of the GAMS objective variable to the quantile level \( \text{theta} \) will be optimized

• If the variable pair \( x \ cVaR_x \) is defined, GAMS will replace its objective variable by \( cVaR_x \), which will become the cVaR of \( x \) to the quantile level \( \text{theta} \)
Combining EV and cVaR

It is also possible optimize a combination of the expected value and the conditional value at risk like this:

### GAMS Code

```gams
... 
defobj.. 
   obj =e= lambda*EV_r + (1-lambda)*CVaR_r;

ExpectedValue r EV_r
cvarlo             r CVaR_r 0.1
...```

---

*Expected Value (EV)*: The expected value is the long-term average of the random variable. It is calculated by summing all possible values, each multiplied by its probability.

*Conditional Value at Risk (cVaR)*: A risk measure that quantifies the expected value of a random variable conditional on the event that it is below a certain threshold (usually a quantile of the loss distribution). It is a measure of the average loss in the worst-case scenarios.
Output Extraction

- The expected value of the solution can be accessed via the regular .L and .M fields.
- In addition, the following information can be stored in a parameter by scenario:
  - `level`: Levels of variables or equations.
  - `marginal`: Marginals of variables or equations.
  - `randvar`: Realization of a random variable.
  - `opt`: Probability of each scenario.
- This needs to be stored in a separate dictionary:
  ```
  Set dict / scen .scenario.''
  bf .randvar .s_bf
  ' .opt .srep
  x .level .s_x/;
  ```
Adding Uncertainty to Transport

... demand(j).. sum(i, x(i,j)) =g= bf*b(j) - U(j); ...

file emp / '%emp.info%'; put emp '* problem %gams.i%'/;
$onput
randvar bf discrete 0.3 0.95
       0.5 1.00
       0.2 1.05

stage 2 bf u demand
$offput
putclose emp;

Set scen scenarios / s1*s3 /;
Parameter
   s_bf(scen,j) demand factor realization by scenario
   s_x(scen,i,j) shipment per scenario
   s_s(scen);

Set dict / scen .scenario."
   bf .randvar .s_bf
   x .level .s_x /;

Solve transport using emp minimizing z scenario dict;
Summary
## Available GAMS SP Solvers

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<th>DE</th>
<th>DECIS</th>
<th>LINDO</th>
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<tr>
<td>randVar (parametric)</td>
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Conclusion

- Deterministic examples from all kind of application areas exist already (e.g. ~400 in the GAMS Model Library)

- Easy to add uncertainty to existing deterministic models, to
  - ... either use specialized algorithms (DECIS, LINDO)
  - ... or create Deterministic Equivalent and select from wide range of existing GAMS solver links (DE, free)

- New SP examples in the GAMS EMP Library

- More work to be done:
  - Scenario tree support
  - Sampling
  - ...
## Contacting GAMS

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<th>Europe</th>
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