Mathematical Optimization in Finance: Closing the gap

Franz Nelißen
GAMS Software GmbH

FNelissen@gams.com

GOR 2003, Heidelberg
Overview

- Portfolio Optimization Models
- Modeling Approaches
- Example
- Stochastic Programming
- Summary
Mathematical Optimization in Finance

- Very active research field with significant contributions and important practical applications
- Some of the reasons:
  - Continual stream of challenging problems with obvious impact of uncertainty
  - High availability of data
  - Validation potential – benchmarking
  - Very competitive and liquid markets ➔ Many instruments and strategies
  - $$
Portfolio Optimization Models

- Mean-Variance Analysis
- Portfolio Models for Fixed Income
- Scenario Optimization
- Indexation Models
- Stochastic Programming
Modeling Approaches

- General programming languages
  - C++, Delphi, FORTRAN, Java, VBA, ...
- Spreadsheets
- Algebraic Modeling Languages (AML)
- Mixture of different approaches
Algebraic Modeling Languages (AML)

- **Declarative approach**
  - Implementation of the optimization problem is close to its mathematical formulation:
    - Variables, constraints with arbitrary names
    - Sets, indices, algebraic expressions, powerful sparse index and data handling
    - Efficient but simple syntax
  - Model formulation contains no hints how to process the model
    → AML translates this representation into another form suitable for the optimization algorithm

- Also **procedural elements**: Loops, procedures, macros, …
Design Features

- Separation of model and data
  - Problem structure is data independent

- Separation of model and solution methods
  - Multiple model types and solvers
    - LP, MIP, NLP, QP, QPMIP, MINLP,…
    - Links to various commercial codes and to research codes (open solver interface)

- Models are scalable and platform independent
Example

- Simple Mean Variance Model

\[ \text{Min } \sum_{i=1}^{I} \sum_{j=1}^{J} x_i Q_{i,j} x_j \]

\[ \text{s.t. } \sum_{i=1}^{I} \mu_i x_i \geq M \]

\[ \sum_{i=1}^{I} x_i = 1, \quad x_i \geq 0 \]

- GAMS Formulation:

```
  vbal..  v =e= sum((i,j), x(i)*q(i,j)*x(j));
  mbal..  sum(i, mu(i)*x(i)) =g= M;
  budget.. sum(i, x(i)) =e= 1;
  x.lo(i) = 0;
```
$eolcom #
Set i analyzed investments;
Alias (i,j);
Parameter mu(i) expected return,
q(i,j) covariance matrix,
mup target expected return for the portfolio;
Variables v variance of portfolio,
M mean return of the portfolio,
x(i) fraction of the portfolio that consists of i;
Equations vbal variance definition,
mbal mean balancing constraint,
budget budget constraint;
vbal.. v =e= sum((i,j), x(i)*q(i,j)*x(j));
mbal.. sum(i, mu(i)*x(i)) =g= M;
budget.. sum(i, x(i)) =e= 1;
$include data.inc # get data from external file
* some bounds
M.lo   =  mup;  # target return of the portfolio
x.lo(i) = 0;  # no short selling

Model var1 / vbal, mbal, budget / ;
Solve var1 minimizing v using nlp ;
display v.l, M.l, mup, x.l;
Example - Data

Set i / cn, fr, gr, jp, sw, uk, us /
Parameter  mu(i) /
    cn  0.1287,  fr  0.1096,  gr  0.0501,   jp     0.1524, 
    sw  0.0763,  uk  0.1854,  us  0.0620                   /;
Table  q(i,j) 
          cn   fr   gr   jp   sw   uk   us  
    cn  42.18 
    fr  20.18  70.89 
    gr  10.88  21.58  25.51 
    jp  5.30   15.41  9.60  22.33 
    sw  12.32  23.24  22.63 10.32  30.01 
    uk  23.84  23.80  13.22 10.46  16.36 42.23 
    us  17.41  12.62  4.70  1.00  7.20  9.90  16.42 ;
Scalar  mup   / 0.115 /;
q(i,j)$$(ord(j) gt ord(i)) = q(j,i) ;
Example - Solution

VARIABLE v.L = 10.487  variance of the portfolio
VARIABLE m.L = 0.115   mean return of the portfolio
PARAMETER mup = 0.115  target expected return for the portfolio

VARIABLE x.L  fraction of the portfolio that consists of security i
  gr 0.014,  jp 0.465,  uk 0.090,  us 0.430
Example - Extensions

- Short Sales
- Efficient Frontiers
- Trading Restrictions (“Zero or Range“ – Constraints)
  → MINLP Problem
Table bdata(i,pd) portfolio data and trading restrictions

<table>
<thead>
<tr>
<th></th>
<th>old</th>
<th>umin</th>
<th>umax</th>
<th>lmin</th>
<th>lmax</th>
</tr>
</thead>
<tbody>
<tr>
<td>cn</td>
<td>0.2</td>
<td>0.03</td>
<td>0.11</td>
<td>0.02</td>
<td>0.30</td>
</tr>
<tr>
<td>fr</td>
<td>0.2</td>
<td>0.04</td>
<td>0.10</td>
<td>0.02</td>
<td>0.15</td>
</tr>
<tr>
<td>gr</td>
<td>0</td>
<td>0.04</td>
<td>0.07</td>
<td>0.04</td>
<td>0.10</td>
</tr>
<tr>
<td>jp</td>
<td>0</td>
<td>0.03</td>
<td>0.11</td>
<td>0.04</td>
<td>0.10</td>
</tr>
<tr>
<td>sw</td>
<td>0.2</td>
<td>0.03</td>
<td>0.20</td>
<td>0.04</td>
<td>0.10</td>
</tr>
<tr>
<td>uk</td>
<td>0.2</td>
<td>0.03</td>
<td>0.10</td>
<td>0.04</td>
<td>0.15</td>
</tr>
<tr>
<td>us</td>
<td>0.2</td>
<td>0.03</td>
<td>0.10</td>
<td>0.04</td>
<td>0.30</td>
</tr>
</tbody>
</table>
Trading Restrictions – Formulation

Variables

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i(i)$</td>
<td>fraction of portfolio increase</td>
</tr>
<tr>
<td>$x_{d}(i)$</td>
<td>fraction of portfolio decrease</td>
</tr>
<tr>
<td>$y(i)$</td>
<td>binary switch for increasing current holdings of $i$</td>
</tr>
<tr>
<td>$z(i)$</td>
<td>binary switch for decreasing current holdings of $i$</td>
</tr>
</tbody>
</table>

Binary Variables

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$, $z$</td>
<td>positive variables $x_i$, $x_{d}$</td>
</tr>
</tbody>
</table>

Equations

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{maxinc}(i)$</td>
<td>bound of maximum lot increase of fraction of $i$,</td>
</tr>
<tr>
<td>$\text{mininc}(i)$</td>
<td>bound of minimum lot increase of fraction of $i$,</td>
</tr>
<tr>
<td>$\text{maxdec}(i)$</td>
<td>bound of maximum lot decrease of fraction of $i$,</td>
</tr>
<tr>
<td>$\text{mindec}(i)$</td>
<td>bound of minimum lot decrease of fraction of $i$,</td>
</tr>
<tr>
<td>$\text{binsum}(i)$</td>
<td>restrict use of binary variables,</td>
</tr>
<tr>
<td>$\text{xdef}(i)$</td>
<td>final portfolio definition;</td>
</tr>
</tbody>
</table>

Model

```
xdef(i).. x(i) =e= bdata(i,'old') - xd(i) + xi(i);
maxinc(i).. xi(i) =l= bdata(i,'umax')* y(i);
mininc(i).. xi(i) =g= bdata(i,'umin')* y(i);
maxdec(i).. xd(i) =l= bdata(i,'lmax')* z(i);
mindec(i).. xd(i) =g= bdata(i,'lmin')* z(i);
binsum(i).. y(i) + z(i) =l= 1;
```

Model var2 /all/; Solve var2 minimizing v using minlp ;
VARIABLE \( v.L = 15.418 \) variance of the portfolio
VARIABLE \( m.L = 0.115 \) mean return of the portfolio
PARAMETER \( mup = 0.115 \) target expected return for the portfolio

---- 147 PARAMETER report summary report

<table>
<thead>
<tr>
<th></th>
<th>Increase</th>
<th>Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>old</td>
<td>new</td>
</tr>
<tr>
<td>cn</td>
<td>0.200</td>
<td>0.056</td>
</tr>
<tr>
<td>fr</td>
<td>0.200</td>
<td>0.050</td>
</tr>
<tr>
<td>gr</td>
<td>0.070</td>
<td>0.070</td>
</tr>
<tr>
<td>jp</td>
<td>0.110</td>
<td>0.110</td>
</tr>
<tr>
<td>sw</td>
<td>0.200</td>
<td>0.122</td>
</tr>
<tr>
<td>uk</td>
<td>0.200</td>
<td>0.292</td>
</tr>
<tr>
<td>us</td>
<td>0.200</td>
<td>0.300</td>
</tr>
</tbody>
</table>
Recent Developments

- Support of codes, which take advantage of special problem structures:
  - QP / QPMIP via Interior Point Methods
  - Conic Programming
- Support of various global optimization codes
Stochastic Programming (SP)

- Models so far are static: Decision is made, then not further modified
- Stochastic Programming models allow sequence of decisions
- Elements:
  - **Scenarios**: Complete set of possible discrete realizations of the uncertain parameters with probabilities
  - **Stages**: Decisions points. First stage decisions now, second stage decision (depending of the outcome of the first stage decision) after a certain period and so on
  - **Recourse**: Describes how decision variables can adapt to the different outcomes of the random parameters at each stage
A simple Scenario Tree
A more complex Scenario Tree
Challenges

- Deterministic equivalent: Includes all scenarios and stages of SP
  - Size of model explodes
- Challenges (among others):
  - Generation difficult
  - Solution may not be possible
  - Interpretation and validation of results
- Less applications of SP than one may expect
But: *Number of uncertain parameters is small*

- Efficient representation of the uncertain data within the AML?
- Some scenarios only differ slightly → Can we reduce the number of scenarios?
- Problems are structured → How can we take advantage of the different specialized solution techniques (Decomposition) for SP’s
SP & AML

- Representation of uncertain data structures → New language elements necessary:
  - Special expressions and conventions for stages and scenario trees
  - Random distributions for some problem data
- Support of scenario reduction techniques can dramatically reduce the size of deterministic equivalent
- Automatic translation of problem description into input format for various SP-solvers (OSL SE, DECIS, …)
  → But: Different approaches, not yet clear which standard will be adopted
Summary

- Finance is a success story for OR applications
- Large classes of problems can be solved without major problems
- AML provide a powerful and flexible framework for these classes of models
- Stochastic programming still challenging but a lot of promising developments