Approximation of Non-linear Functions in Mixed Integer Programming

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Joint work with Markus Möller and Susanne Moritz
Outline

1. Non-linear Functions in MIPs
   - design of sheet metal
   - gas optimization
   - traffic flows

2. Modelling Non-linear Functions
   - with binary variables
   - with SOS constraints

3. Polyhedral Analysis

4. Computational Results
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Design of Transport Channels

Goal
Maximize stiffness

Subject To
- Bounds on the perimeters
- Bounds on the area(s)
- Bounds on the centre of gravity

Variables
- topology
- material

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Optimization of Gas Networks

Goal
Minimize fuel gas consumption

Subject To
- contracts
- physical constraints
Gas Network in Detail
Gas Networks: Nature of the Problem

- Non-linear
  - fuel gas consumption of compressors
  - pipe hydraulics
  - blending, contracts

- Discrete
  - valves
  - status of compressors
  - contracts
Pressure Loss in Gas Networks

\[ \frac{\partial x p}{\partial t} + \frac{\lambda \rho_o \rho_o T}{2DA^2 z_o T_o} \frac{|q| q z}{p} \]

\[ + \frac{\rho_o \rho_o T}{A^2 z_o T_o} \partial_x \left( \frac{q^2 z}{p} \right) + \frac{\rho_o}{A} \partial_t q \]

\[ + g \rho_o z_o T_o \frac{q}{p} \partial_x \left( \frac{hp}{z} \right) = 0 \]

horizontal pipes → stationary case

\[ p_{out}^2 = p_{in}^2 - ff |q|q \]
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Approximation of Pressure Loss: Binary Approach

\[ \lambda^i \geq 0, \text{ for grid point } i \in \Lambda \]
\[ y^i \in \{0, 1\}, \text{ for triangle } j \in Y \]

(1) \[ \sum_{i \in \Lambda} \lambda^i = 1 \]

(2) \[ \sum_{j \in Y} y^j = 1 \]

(3) \[ y^j \leq \sum_{k \in N^j} \lambda^k \]

(4) \[ p_{in} = \sum_{i \in \Lambda} p_{in}^i \lambda^i \]

(5) \[ p_{out} = \sum_{i \in \Lambda} p_{out}^i \lambda^i \]

(6) \[ q = \sum_{i \in \Lambda} q^i \lambda^i \]
Approximation of Pressure Loss: SOS Approach

\[
\begin{align*}
\pi_{in} &= \sum_{i \in \Lambda} p_{in}^i \lambda^i \\
q &= \sum_{i \in \Lambda} q^i \lambda^i \\
p_{out} &= \sum_{i \in \Lambda} p_{out}^i \lambda^i \\
\sum_{i \in \Lambda} \lambda^i &= 1 \quad (*) \\
\lambda^i &\geq 0
\end{align*}
\]

\( (*) \) must meet the triangle condition
Branching on SOS Constraints

\[ \sum \lambda_i = 1 \]

\[ \lambda_i = \frac{1}{3} \]

\[ \sum \lambda_i = 1 \]
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The SOS Constraints: General Definition

Given

- grid points $\Lambda = \{1, \ldots, n\}$ and
- a set of subsets $Y = \{N^1, \ldots, N^d\}$, $N^i \subseteq \Lambda$.

A vector $\lambda$ satisfies the set condition for $Y$ and

$$\sum_{i \in \Lambda} \lambda^i = 1 \quad (*)$$

if $\{i \in \Lambda : \lambda^i > 0\} \subseteq N^r$ for some $r$.

$(*)$ is called **SOS constraint of Type $k$**, where

$$k = \max_i |N^i|.$$
The SOS Constraints: Special Cases

- **SOS Type 2 constraints**
  \[ N^i = \{ i, i + 1 \} \]

- **SOS Type 3 constraints**
  \[ |N^i| = 3 \quad \forall i \]
The Binary Polytope

Let $P_k = \text{conv} \{ \lambda \in \mathbb{R}_+^\Lambda, y \in \{0, 1\}^Y :$

\begin{align*}
(1) \quad & \sum_{i \in \Lambda} \lambda^i = 1 \\
(2) \quad & \sum_{j \in Y} y^j = 1 \\
(3) \quad & y^j \leq \sum_{i \in N^j} \lambda^i \}
\end{align*}

with $k = \max_j |N^j|$
The Binary Polytope: Inequalities

For $\emptyset \neq J \subset Y$ and $I := \bigcup_{j \in J} N^j$ let

$$\sum_{j \in J} y^j \leq \sum_{i \in I} \lambda^i \quad (*)$$

Theorem. (*) describes $P_2$ and $P_3$ completely.
The SOS Polytope

\[ P_\Delta = \left\{ \left( \frac{\lambda_1}{\lambda_2} \right) \in \mathbb{R}^{|\Lambda_1| + |\Lambda_2|} \mid \sum_{j \in \Lambda_1} \lambda_1^j = 1, \sum_{j \in \Lambda_2} \lambda_2^j = 1, \sum_{j \in \Lambda_1} p_1^j \lambda_1^j - \sum_{j \in \Lambda_2} p_2^j \lambda_2^j = 0, \lambda_1^j, \lambda_2^j \geq 0 \right\} \]

\lambda_1, \lambda_2 \text{ satisfy the set condition for } Y_1 \text{ and } Y_2 \}.
## The SOS Polytope: Increasing Complexity

| $|\Delta|$ | $|Y|$ | Vertices | Facets | Max. Coeff. |
|-------|------|---------|-------|------------|
| 8     | 12   | 16      | 18    | 25         |
| 16    | 18   | 49      | 47    | 42         |
| 24    | 24   | 73      | 90    | 670        |
| 32    | 32   | 142     | 10492 | 50640      |
Theorem. There exist only polynomially many vertices \( v_1, \ldots, v_l \) with \( l \leq 9|Y_1| |Y_2| \).

- The vertices can be determined algorithmically.
- This yields a polynomial separation algorithm by solving for given \( \lambda_1^* \) and \( \lambda_2^* \):

\[
\max \quad a^T \left( \begin{array}{c}
\lambda_1^* \\
\lambda_2^*
\end{array} \right) - \alpha \\
\text{s.t.} \quad a^T v_i \leq \alpha \quad \text{for} \quad i = 1, \ldots, k
\]
The SOS Polytope: Generalizations

• Pipe to pipe with respect to pressure and flow
• Several pipes to several pipes
• Pipes to compressors (SOS constraints of Type 4)
• General Mixed Integer Programs:

Consider $Ax=b$ and a set $I$ of SOS constraints of Type $k_i$ for $i \in I$ such that each variable is contained in exactly one SOS constraint. If the rank of $A$ (incl. $I$) and $\max_i k_i$ are fixed then

$$P = \text{conv} \left\{ x \in \mathbb{R}^n \mid Ax = b \right\}$$

$x$ satisfies the set condition for $i \in I$

has only polynomial many vertices.
Binary versus SOS Approach

- **Binary**
  - more (binary) variables
  - more constraints
  - complex facets
  - LP solutions with fractional $y$ variables and correct $\lambda$ variables

- **SOS**
  + no binary variables
  + triangle condition can be incorporated within branch & bound
  + underlying polyhedra are tractable
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Computational Results

<table>
<thead>
<tr>
<th>Nr of Pipes</th>
<th>Nr of Compressors</th>
<th>Total length of pipes</th>
<th>Time ((\varepsilon = 0.05))</th>
<th>Time ((\varepsilon = 0.01))</th>
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<tbody>
<tr>
<td>11</td>
<td>3</td>
<td>920</td>
<td>1.2 sec</td>
<td>2.0 sec</td>
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<td>3</td>
<td>1200</td>
<td>1.2 sec</td>
<td>9.9 sec</td>
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<td>31</td>
<td>15</td>
<td>2200</td>
<td>11.5 sec</td>
<td>104.4 sec</td>
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