Mixed Complementarity Formulations of Stochastic Equilibrium Models with Recourse

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GOR Workshop “Optimization under Uncertainty”
Bad Honnef, Germany, October 20-21, 2005
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Three illustrative models

- Lifecycle consumption-savings decisions with income uncertainty (finite horizon NLP)
- Ramsey growth model with uncertain technology change (infinite-horizon NLP)
- Growth model with anticipated tax policy and parameter uncertainty (infinite-horizon MCP)
Programming Techniques

• Event tree management logic tool (probtree)
• Tight formulations
• Graphical tools for debugging (treeplot) and reporting (fanplot)
• Complementarity programming in a stochastic framework
Mixed Complementarity Problem - MCP Definition

Given: \( F : R^n \to R^n \), \( \ell, u \in R^n \)

Find: \( z, w, v \in R^n \)

such that \( F(z) = w - v \)
\( \ell \leq z \leq u, \ w \geq 0, \ v \geq 0 \)
\( w^T(z - \ell) = 0 \), \( v^T(u - z) = 0 \).
Credits

• DMR School
  – Sherman Robinson, Jaime deMelo, Kemal Dervis

• MPSGE School
  – Tom Rutherford, Alan Manne

• MCP Solvers
  – MILES Tom Rutherford
  – PATH Michael Ferris, Stephen Dirkse, Todd Munson
Lifecycle Model

- A lifecycle savings investment model in which there is income uncertainty maximizing the discounted expected utility
- Utility function: Logarithm of consumption
- Version 1: Borrowing and savings
- Version 2: Only savings
Tree Plot of Lifecycle Probability Tree

gams treeplot
Transition in GAMS

sets
t   Time periods for a typical life-cycle (ages) /20*75/

sw   States of world / 30pa peak earn  30K per year
     40pa peak earn  40K per year
     80pa peak earn  80K per year
     100pa peak earn 100K per year /

transition(t,sw,sw) State transitions /
   20.40pa. 30pa Learn at age 20 if you are going to college
   25.40pa. 80pa Learn at age 25 if you earn a PhD degree
   35.80pa.100pa Learn at age 35 that you are good at business /

parameter pi(sw) Subjective probability /
   30pa 0.3
   40pa 0.4
   80pa 0.2
   100pa 0.1 /
Tree Plot of Lifecycle Probability Tree

gams treeplot --piscale=yes
sets
  eq(t,sw) Equilibrium structure: sw is active in t
  st(t,sw,sow) State transitions: sw transitions to sow in t
  sm(t,sw,sow) State matching: sw is represented by sow in t
  sp(t,sw) Preceding state of world

* Call Tom's probtree utility
$batinclude probtree t sw transition eq st sm

loop(st(t,sow,sw), sp(t,sw) = ord(sw)-ord(sow));

b.lo(t,sw)=-inf; b.up(t,sw)=+inf; // Borrowing and saving
b.fx(tlast,sw) = 0; // No debt or savings at the end

solve lcycle using nlp maximizing eu; // solve stoch. model

loop(sw, pi(sow)=0; pi(sw)=1; // Solve deterministic for sw
solve lcycle using nlp maximizing eu );
Live Runs - Lifecycle Model
Consumption Profiles with Liquidity Constraints

Lifecycle Consumption-Savings Model
Asset Balances with Borrowing

Lifecycle Consumption-Savings Model

![Graph showing asset balances with borrowing over time.]
Asset Balances with Liquidity Constraints

Lifecycle Consumption-Savings Model
1. Event tree representation via state transitions using **probtree** utility
2. Event tree visualization using **treeplot** utility.
3. Without hedging possibilities stochasticity cannot be exploited
4. Importance of visual presentation of input and output
Classic Ramsey Model

- Ramsey Model with uncertain technology change
- Infinite horizon model in which there is an uncertain transition path to a deterministic horizon
- Motivate formulations with and without non-anticipativity constraints

The Ramsey Model

implicit algebra

\[
\max E(\sum_{t=0}^{\infty} \beta_t u(\tilde{c}_t))
\]

s.t.

\[
\tilde{c}_t + \tilde{i}_t = \tilde{y}_t \\
\tilde{y}_t = f(\tilde{k}_t) \\
\tilde{k}_{t+1} = (1 - \delta)\tilde{k}_t + \tilde{i}_t \\
\tilde{i}_t \geq 0
\]
Data Structure for Stochastic Programming

\( s \) Set of scenarios, associated with all leaves at the bottom of the event tree.

\( t \) Set of time periods in the model. There must be at least as many time periods in the model as there are levels in the event tree.

\( \Sigma_t \) Each node in the event tree is labelled with a scenario index \( s \). Set \( \Sigma_t \) defines the set of “active scenarios” in time period \( t \). If a scenario has appeared in time period \( t \), it must be present in all subsequent time periods:

\[ s \in \Sigma_t \Rightarrow s \in \Sigma_{\tau} \forall \tau > t \]
\( \Gamma_{st} \) Defines the scenarios which branch on from scenario \( s \) in time period \( t \). Once a scenario has “appeared”, it must follow itself in all subsequent periods, i.e.:

\[
s \in \Sigma_t \Rightarrow s \in \Gamma_{st} \forall \tau \geq t
\]

\( \Omega_{ts} \) Defines the set of scenarios which branch from scenario \( s \) or from a “descendent of \( s' \)” in time period \( t \) or in any later period:

\[
s' \in \Omega_{st} \Rightarrow \exists t' \geq t \text{ such that } s' \in \Gamma_{st'}
\]
Ramsey Model
Conventional Explicit Syntax

$$\max \sum_s \pi_s \left( \sum_{t=0}^{\infty} \beta^t u(c_{st}) \right)$$

s.t.

$$c_{st} + i_{st} = y_{st}$$
$$y_{st} = f(k_{st})$$
$$k_{s',t+1} = (1 - \delta)k_{st} + i_{st} \quad \forall s' \in \Gamma_{st}$$

and non-anticipativity constraints:

$$\begin{cases} 
c_{s'\tau} = c_{st} \\
k_{s'\tau} = k_{st} \\
y_{s'\tau} = y_{st} \\
i_{s'\tau} = i_{st}
\end{cases} \quad \forall s' \in \Omega_{st}, \tau \leq t$$
Ramsey Model
Tight Formulation

\[
\max \sum_s \pi_s \left( \sum_{t=0}^{\infty} \sum_{s'|s' \in \Omega_{s,t}} \beta_t u(c_{s',t}) \right)
\]

s.t.

\[
c_{st} + i_{st} = y_{st} \quad \forall s \in \Sigma_t
\]

\[
y_{st} = f(k_{st}) \quad \forall s \in \Sigma_t
\]

\[
k_{s',t+1} = (1 - \delta) k_{st} + i_{st} \quad \forall s' \in \Gamma_{st}, \forall s \in \Sigma_t
\]

Note that the non-anticipativity constraints are unnecessary.
Recovery of Solution from a tight Formulation

\[
\begin{align*}
    c'_{S,T} &\leftarrow c_{ST} \\
    k'_{S,T} &\leftarrow k_{ST} \\
    y'_{S,T} &\leftarrow y_{ST} \\
    i'_{S,T} &\leftarrow i_{ST}
\end{align*}
\]

\[\forall s' \in \Omega_{ST}, \tau \leq t\]
Assumed Probability Distribution for Date of Technical Change

Ramsey Growth Model
Cumulative Probability Density Function

![Cumulative Probability Density Function Graph](image-url)
set t   Time periods in the model /2005*2100/  
    sw States of world /2010*2030,never/  
transition(t,sw,sw) State transitions;

* Define the event tree by specifying which states  
* generate transitions. In this case 2010 defines the root  
* node of the tree.

transition("2010","2010","2011") = yes;  
loop((t,sw)$transition(t,sw,sw+1),  
     transition(t+1,sw+1,sw+2) = yes);

pi(sw) = exp(-var*sqr(ord(sw)-card(sw)/2)); // Normal distribution

• Normalize:

pi("never") = 0; pi(sw) = 0.75*pi(sw)/sum(sow,pi(sow));  
pi("never") = 1 - sum(sw,pi(sw));
Live Runs – Ramsey Model

```plaintext
$title Liquidity-Constrained Lifecycle Expenditure
$call ra merged.gdx
$if exist merged.gdx $goto start
$call gams deter --sw=30d gdx=30d --nopt=yes
$call gams deter --sw=40d gdx=40d --nopt=yes
$call gams deter --sw=80d gdx=80d --nopt=yes
$call gams deter --sw=100d gdx=100d --nopt=yes
$call "gdxmerge 30d.gdx 40d.gdx 80d.gdx 100d.gdx"
$label start
set t Time periods for
tr(t) Retirement period
tl(t) Labels for plots
l(t) Last period
sw States of world /
transition(t,sw) State trans
transition(t,sw) States of world /
states(t,sw,sw) States of world
execute unload 'probtree.gdx', , , transition;
execute 'gams probtree.gdx=treepeople.gdx';
alias (sw,states,states);
set eq(t,sw) Equilibrium structure (sw
st(t,sw,sw) State transitions (sw tran
sm(t,sw,sw) State matching (sw is rep
execute_load 'treepeople.gdx', eq, st, sm;
```
Equilibrium Wage Rates

Ramsey Growth Model
Lessons Learned

1. Two mathematically equivalent formulations: with and without non-anticipativity constraints (NAC). Tight formulation is preferred (no NAC) with easy recovery of solution of the explicit problem (using Tom‘s tools)

2. Construction of transition matrix

3. Discretization of continuous distributions
Enhanced Ramsey Model

- Complementarity problem based on the Ramsey model
- Features an ad-valorem tax on capital services, hence there is no corresponding optimization problem
- In this application the policy is deterministic - a capital tax is applied five years in the future.
• The uncertainty concerns a model parameter - the capital-labor elasticity of substitution.

• The investors in the model only know the probability distribution of this parameter, the true value of which will only be revealed after the policy shock.

• We use the model to characterize a rational-expectations forecast of the impact of the capital tax, taking into account hedging behavior which reflects uncertainty regarding model parameters.
Stochastic Structures

set t       Time periods in the model /2005*2060/
sw         States of world /s0*s10/
transition(t,sw,sw) State transitions;

transition("2008","s0",sw)   = yes;
transition("2008","s0","s0") = no;

parameter pi(sw) Perceived probability;
pi(sw) = 1/card(sw);
Explicit MCP Formulation
Formulation using MPSGE Sublanguage

$model:stc$

$sectors:

\begin{align*}
  &u(sw) \quad ! \text{Utility} \\
  &y(t,sw) \quad ! \text{Output} \\
  &i(tp,sw) \quad ! \text{Investment} \\
  &k(tp,sw) \quad ! \text{Capital} \\
  &c(t,sw) \quad ! \text{Consumption} \\
\end{align*}

\ldots

$prod:y(t,sw) \quad s:sigma(sw)$

\begin{align*}
  &o:p(t,sw) \quad q:y0 \\
  &i:pl(t,sw) \quad q:10 \\
  &i:rk(t,sw) \quad q:rk0 \quad a:ra \quad t:tk(t)
\end{align*}
Live Runs
Ramsey MCP Model

```gams
$title Liquidity-Constrained Lifecycle Expenditure
$call rm merged.gdx
$if exist merged.gdx $goto start
$call gams dete --sw=30d.gdx=30d --noplot=yes
$call gams dete --sw=40d.gdx=40d --noplot=yes
$call gams dete --sw=80d.gdx=80d --noplot=yes
$call gams dete --sw=100d.gdx=100d --noplot=yes
$call gdxmerge 30d.gdx 40d.gdx 80d.gdx 100d.gdx

$label start

set 
   t            Time periods for transition
   tr(t)        Retirement period
   tl(t)        Labels for plots
   latt(t)      Last period
   sw           States of world /
transition(t,sw,sw') State transition
 20.40pa,30pa, 
 25.40pa,80pa, 
 35.80pa,100pa /;

execute_unload 'probtree.gdx', t, sw, transition;

execute 'gams probtree.gdx=tree logic.gdx';

alias (sw,sw',sw'');

set 
   eq(t,sw)      Equilibrium structure (sw)
   st(t,sw,sw')  State transitions (sw tran
   sm(t,sw,sw')  State matching (sw is rep

execute_load 'tree logic.gdx', eq, st, sm;
```

---

```gams
21 3 1.26402977658E+02 4.5E-04 18 2
26 4 1.2641233592E+02 6.28E-04 25 1
31 4 1.2643481968E+02 4.4E-04 28 1
36 4 1.2643490972E+02 1.3E-04 33 3
39 4 1.2643492457E+02 5.9E-09 34

** Optimal solution. Reduced gradient less than 1e-6.

---

1 cycle.gdx (9) 2 Mb
1 cycle.gdxmerge 30d.gdx 40d.gdx 80d.gdx 100d.gdx
Reading file: 30d.gdx
Reading file: 40d.gdx
Reading file: 80d.gdx
Reading file: 100d.gdx
Merge complete
1 cycle.gdx (81) 3 Mb
plot.gdx (79) 7 Mb
1 cycle.gdx (124) 3 Mb
GDXin=C:\work\alex\lifecycle\merged.gdx
1 cycle.gdx (167) 3 Mb
plot.gdx (306) 3 Mb
1 cycle.gdx (168) 3 Mb
plot.gdx (306) 3 Mb
1 cycle.gdx (169) 3 Mb
plot.gdx (306) 3 Mb
1 cycle.gdx (170) 3 Mb
plot.gdx (306) 3 Mb
1 cycle.gdx (172) 3 Mb
Starting execution
1 cycle.gdx (25) 4 Mb
1 cycle.gdx (34) 4 Mb
GDXin=C:\work\alex\lifecycle\tree logic.gdx
1 cycle.gdx (267) 4 Mb
---
---
---
---
---
---
Return to Capital
Capital Tax Impacts with Structural Uncertainty
Output

Capital Tax Impacts with Structural Uncertainty
Lessons Learned

1. Lessons learned from NLP models carry over to MCP formulation
2. Application specific sublanguage MPSGE naturally accommodates stochastic formulations
3. Fan diagrams allow effective presentation of large number of scenarios
Summary

- Tom, please add some more comments
- The models will be available on www.mpsge.org and the GAMS web site www.gams.com
- Reproducability
- Consistent notation for optimization and complementarity