Random Outerplanar Graphs

Studienarbeit

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Abstract

Let $\mathcal{P}_n$ be the set of labelled planar graphs with n vertices and let $\mathcal{P}_{n,m}$ be the set of labelled planar graphs with n vertices and m edges. Osthus, Prömel, and Taraz proved in [7] that $|\mathcal{P}_n| \le n!\,(37.3)^{n+o(n)}$ and that almost all graphs in $\mathcal{P}_n$ have at most $2.56n$ edges. They counted the number of triangulations a planar graph is contained in to obtain an upper bound of $|\mathcal{P}_{n,m}|$. We will here use similar techniques to the set $\mathcal{P}_n^{\text{out}}$ of labelled outerplanar graphs with n vertices to show that $\left|\mathcal{P}_n^{\text{out}}\right| \le n!\,9^{n+o(n)}$, and that almost all graphs in $\mathcal{P}_n^{\text{out}}$ have at most $1.73n$ edges.

Furthermore we apply techniques from [6] to show that the expected number of isolated vertices of a labelled outerplanar graph lies between $1.29 \cdot 10^{-5} - \varepsilon$ and $\frac{1}{2} + \varepsilon$, $\varepsilon > 0$, and we give asymptotics for the expected number of vertices of degree 1 and 2 in a labelled outerplanar graph.

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Contents

1 Introduction

Random discrete structures play an important role in the design of algorithms on these structures, since an efficient sampling algorithm can be used to generate "typical" test-instances. To write such an algorithm which samples discrete structures according to a given probability distribution it's essential to know, how many of these structures with a given parameter are there and how they can be counted.

Outerplanar graphs are planar graphs with an embedding where every vertex lies on the outer face. Let $\mathcal{P}_n^{\text{out}}$ be the set of labelled outerplanar graphs with vertex set $[n] := \{1, \ldots, n\}$. Bodirsky and Kang have shown in [2] how to generate labelled and unlabelled outerplanar graphs with n vertices uniformly at random in polynomial time in n. They presented a counting technique which is based on the decomposition of a graph according to its block structure, to compute the exact number of labelled outerplanar graphs.

Denise, Vasconcellos, and Welsh have shown in [3] that there is a constant γ_u, the unlabelled planar graph growth constant, such that $u(n)^{1/n} \to \gamma_u$ as $n \to \infty$, where $u(n)$ denotes the number of unlabelled planar graphs on n vertices. McDiarmid, Steger, and Welsh proved in [6] that there is also a labelled planar graph growth constant γ_l such that $(|\mathcal{P}_n| / n!)^{1/n} \to \gamma_l$ as $n \to \infty$, where $\mathcal{P}_n$ denotes the class of labelled planar graphs on n vertices.

They were also studying various properties of a random planar graph, drawn uniformly at random from $\mathcal{P}_n$, e.g. they have shown that the probability that a random planar graph is connected is bounded away from 0 and from 1. Since most of their conclusions were derived from more general results, it is easy to obtain similar results for outerplanar graphs: Let $\mathcal{C}$ be any non-empty finite set of graphs. It is called *addable* if a graph G is in $\mathcal{C}$ if and only if each component of G is in $\mathcal{C}$ and for each graph G in $\mathcal{C}$, if u and v are vertices in distinct components of G then the graph obtained from G by adding an edge joining u and v is also in $\mathcal{C}$. In [6] they used a Markov-chain to show that for a graph R sampled uniformly at random from $\mathcal{C}$, the random number $\kappa(R)$ of components of R is stochastically dominated by $1 + X$, where X has the Poisson distribution with mean 1. Clearly, this theorem can be applied to labelled outerplanar graphs, using $\mathcal{C} = \mathcal{P}_n^{\text{out}}$, so that the probability that a labelled outerplanar graph is connected, is bounded from below by $1/e$ and the expected number of connected components is at most 2.

The set of graphs $\mathcal{C}$ is called *small* if there is a constant d such that the number $f(n)$ of graphs in $\mathcal{C}$ on the vertices $1, \ldots, n$ satisfies $f(n) \leq d^n n!$ for all n sufficiently large. For a non-empty small and addable class $\mathcal{C}$, McDiarmid, Steger, and Welsh have shown in [6] that there is a finite *growth constant* $c > 0$ such that $(f(n)/n!)^{1/n} \to c$ as $n \to \infty$. Since $\mathcal{P}_n^{\text{out}} \subseteq \mathcal{P}_n$ and the class of labelled planar graphs is small, the class of labelled outerplanar graphs is small too, and clearly it is addable. So we can apply the same theorem from [6] to conclude that there is a finite constant γ_l^{out} such that $(|\mathcal{P}_n^{\text{out}}| / n!)^{1/n} \to \gamma_l^{\text{out}}$ as $n \to \infty$.

In Theorem 2.6 we will give an upper bound on $|\mathcal{P}_{n,m}^{\text{out}}|$, the number of labelled outerplanar graphs on n vertices and m edges, and conclude that $\gamma_l^{\text{out}} \leq 9$. Bodirsky, Giménez, Kang, and Noy have shown in [1] that $\gamma_l^{\text{out}} \approx 7.32098$. Combining the latter with our upper bound on $|\mathcal{P}_{n,m}^{\text{out}}|$ allows to give an upper bound on the expected number of edges of an outerplanar graph. Gerke and McDiarmid proved in [4] that almost all labelled outerplanar graphs have at least $\frac{7}{5}n$ edges. We will prove that almost all labelled outerplanar graphs have at most $1.73n$ edges.

McDiarmid, Steger, and Welsh investigated in [6] the expected number I_n of isolated vertices of a labelled planar graph on n vertices: They have shown that the probability that a random labelled planar graph on n vertices has an isolated vertex is bounded below by some number greater zero. The required Theorems 4.1 and 5.1 in [6] can be applied to outerplanar graphs as well, so that the probability that a labelled outerplanar graph has an isolated vertex is at least $\frac{1}{72e^3} \cdot \frac{1}{(\gamma_l^{\text{out}})^2} + o(1) > 0$. Using $\gamma_l^{\text{out}} \leq 7.321$ we obtain $I_n^{\text{out}} \geq 1.29 \cdot 10^{-5} + o(1)$. In [3] it has been shown that $|\mathcal{P}_n| \geq (6n - 15) |\mathcal{P}_{n-1}|$ for $n \geq 5$, so that $I_n \leq \frac{1}{6} + o(1)$. For outerplanar graphs, we can show that $I_n^{\mathrm{out}} \leq \frac{1}{2} + o(1)$, see Corollary 3.6. In addition to these results, it was proved in [6] that if I_n tends to a limit as $n \to \infty$ (this is called the *isolated vertices conjecture*), then $I_n \to \gamma_l^{-1}$ as $n \to \infty$. For outerplanar graphs, the isolated vertices conjecture was proved in [1], so that we can follow $I_n^{\mathrm{out}} \to (\gamma_l^{\mathrm{out}})^{-1}$ as $n \to \infty$.

McDiarmid, Steger, and Welsh additionally analyzed how many vertices of a specific degree and how many number of triangular faces a random planar graph on n vertices typically has (in some embedding). Most of these results seem to be easily adaptable for outerplanar graphs as well.

2 Triangulating outerplanar graphs

Denote by $\mathcal{P}_{n,m}^{\mathrm{out}}$ the set of labelled outerplanar graphs with vertex set $[n]$ and m edges. An outerplanar graph on n vertices contains at most $2n-3$ edges. So, denote by $\mathcal{T}_n^{\mathrm{out}}$ the set of labelled maximal outerplanar graphs, $\mathcal{T}_n^{\mathrm{out}} := \mathcal{P}_{n,2n-3}^{\mathrm{out}}$.

Denote by a *block* of a graph G a maximal connected subgraph H of G such that $H - v$ is connected for every vertex $v \in H$. A graph G is called *biconnected* if it is connected and removing an arbitrary vertex yields a connected graph. So a block is an (inclusion-)maximal biconnected component of G.

2.1 Biconnected outerplanar graphs

It is well known, that a biconnected outerplanar graph on at least 3 vertices contains a unique Hamiltonian cycle ([5]). This Hamiltonian cycle is the same for all outerplanar triangulations of this graph. So we can embed each outerplanar triangulation in the plane such that all vertices lie on the outer face. When we fix the embedding of the graph which has all vertices on the outer face, each outerplanar triangulation has an embedding which contains the fixed embedding of the graph.

Proposition 2.1. *The number of (unrooted) maximal outerplanar graphs on $n \geq 3$ vertices is at least $(n-1)! \cdot 2^{n-4}$. The number of (unrooted) maximal outerplanar graphs on $n \geq 3$ vertices with a fixed labelling of the vertices is at least 2^{n-3}.*

Proof. A maximal outerplanar graph on n labelled vertices is a triangulation of the interior of a circle on n vertices. When the labelling of the vertices is fixed, the number of triangulations of such a circle is given by the *Catalan number* $\mathrm{Cat}\,(n)$. It is known (e.g. [8]) that

$$\mathrm{Cat}\,(n) = \frac{1}{n-1}\binom{2n-4}{n-2} \geq 2^{n-3}.$$

When we do not fix the labelling of the Hamiltonian cycle, the number of maximal outerplanar graphs on n vertices is

$$\mathrm{Cat}\,(n) \cdot \frac{(n-1)!}{2} \geq (n-1)! \cdot 2^{n-4}.$$ □

Lemma 2.2. *Every biconnected outerplanar graph $G \in \mathcal{P}_{n,m}^{\mathrm{out}}$ is contained in at least 2^{2n-m-3} outerplanar triangulations.*

Proof. Let G be a biconnected outerplanar graph.

If $n \geq 3$, G contains a unique Hamiltonian cycle and we can fix the embedding that has all vertices on the outer face. We proceed similarly to 3-connected plane graphs (Proposition 5 in [7]): It is sufficient to triangulate each inner face of G independently. The number of triangulations of an inner face (which is always bounded by a cycle, if $n > 3$) is given by the Catalan number $\mathrm{Cat}\,(l)$, where l is the length of the bounding cycle of f. It holds that $\mathrm{Cat}\,(l) \geq 2^{l-3}$ for any $l \geq 3$.

Now denote by l_j the length of the bounding cycle of the j-th inner face. Since every edge lies on the boundary of two faces and the outer face is bounded by n edges, we have $\sum_j l_j = 2m - n$. From Euler's Formula, there are $1-n+m$ inner faces. So the number of outerplanar triangulations on n vertices which contain G is at least

$$\prod_j \mathrm{Cat}\,(l_j) \geq \prod_j 2^{l_j-3} = 2^{\sum_j (l_j-3)} = 2^{2m-n-3(1-n+m)} = 2^{2n-m-3}$$

where the product (and sum) is over all inner faces of G.

If $n = 1$ or $n = 2$, the formula remains true: For $n = 1$, m is zero, so $2^{2n-m-3} = 2^{-1} \leq 1$. For $n = 2$, m equals one, so $2^{2n-m-3} = 2^0 = 1$. □

2.2 Connected outerplanar graphs

Two outerplanar graphs $G_0, G_1 \in \mathcal{P}_n^{\mathrm{out}}$ are called *incomparable* if there is no outerplanar graph $H \in \mathcal{P}_n^{\mathrm{out}}$ containing both G_0 and G_1 as subgraph. Observe that if $G_0 \subseteq H_0$, $G_1 \subseteq H_1$, and G_0 and G_1 are incomparable, then also H_0 and H_1 are incomparable. Observe further that biconnected outerplanar graphs G_0, G_1 whose (unique) Hamiltonian cycles differ are incomparable.

A *triangulation tree* of an embedded outerplanar graph G is a rooted tree whose vertices correspond to embedded outerplanar graphs (with vertex set $[n]$) containing G; whose root corresponds to G; whose leaves correspond to biconnected outerplanar graphs which are incomparable; for any vertex v of the tree, the graphs corresponding to the children of v all have the same number of edges and finally, if a child has t more edges than its parent v, then v has at least 2^t children unless it is an *exceptional vertex*, in which case it still has at least 2^{t-1} children.

Lemma 2.3. *Consider an embedded outerplanar graph $G \in \mathcal{P}_{n,m}^{\mathrm{out}}$ consisting of two blocks H_1 and H_2 which have exactly one vertex x in common.*

Then there exists a triangulation tree of G with no exceptional vertices, unless G is a path on 3 vertices, in which case the root of the triangulation tree is exceptional.

Proof. Let H_1 have at least as many vertices as H_2.

If H_2 has only one vertex, $H_2 = x$, we have $G = H_1$, so G is already biconnected. Hence, let H_2 have at least 2 vertices. Since H_1 has at least as many vertices as H_2, H_1 has at least 2 vertices too.

1. If H_1 has only 2 vertices x and y, H_2 has at least and at most 2 vertices, by assumption. So we are in the case that G is a path on 3 vertices. Adding the remaining edge leads to a biconnected graph G_1. Connecting G with G_1 builds a triangulation tree with exceptional vertex G as root.

2. Let H_1 have at least 3 vertices. Since H_1 is biconnected, it has a unique Hamiltonian cycle Q_1. Let y_1 and y_2 be the neighbours of x at Q_1 ($y_1 \neq y_2$). Denote by P_1 the inner vertices of the path between y_1 and y_2 of Q_1 which does not contain x. (So, $Q_1 = y_1P_1y_2x$, P_1 can be empty). We need to distinguish between two cases for H_2 now.

   (a) The *lollipop-graph*: If H_2 contains only 2 vertices x and z, obtain G_1 by adding the edge y_1z to G and G_2 by adding the edge y_2z to G. Thus, G_1 has the Hamiltonian cycle $y_1P_1y_2xz$ and G_2 the Hamiltonian cycle $y_1P_1y_2zx$.

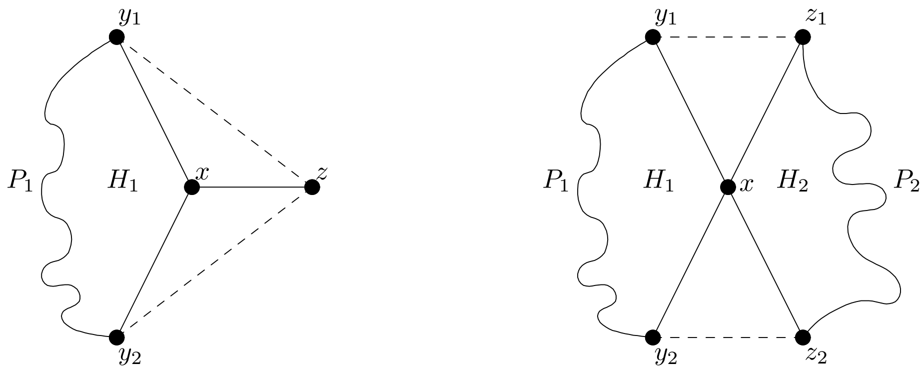

Figure 2.1: The lollipop graph and the butterfly graph from the proof of Lemma 2.3.

(b) The *butterfly-graph*: In the general case, H_2 has at least 3 vertices and is biconnected. Then also H_2 contains a unique Hamiltonian cycle Q_2. Let z_1 and z_2 be the neighbours of x at Q_2 ($z_1 \neq z_2$). Denote by P_2 the inner vertices of the path between z_1 and z_2 of Q_2 which does not contain x. (So, $Q_2 = z_1P_2z_2x$, P_2 can be empty).

Now we merge the Hamiltonian cycles Q_1 and Q_2 to a Hamiltonian cycle of G after adding an edge to G. The first way is to add the edge y_1z_1 to obtain $G_1 := G + y_1z_1$ with Hamiltonian cycle $y_1P_1y_2xz_2P_2z_1$. The second way is to add the edge y_2z_2 to obtain $G_2 := G + y_2z_2$ with Hamiltonian cycle $y_1P_1y_2z_2P_2z_1x$. ☐

Theorem 2.4. *Every connected outerplanar graph $G \in \mathcal{P}^{\text{out}}_{n,m}$ is contained in at least 2^{2n-m-3} labelled outerplanar triangulations, except if G is a tree (on at least 3 vertices). In the latter case, G is still contained in at least $2^{2n-m-4} = 2^{n-3}$ labelled outerplanar triangulations.*

Proof. If G is biconnected, we can apply Lemma 2.2. If G is connected, but not biconnected, then it has at least 3 vertices.

Let B be a block of G which maximizes the number of vertices (in it) and let $H \neq B$ be a block of G which has only the vertex x in common with B (and has at least 2 vertices). Applying Lemma 2.3 to $B + H$, we obtain a triangulation tree of $B + H$.

If the root of this triangulation tree is exceptional, $B + H$ must have been a path on 3 vertices. Since B was chosen to maximize the number of vertices, all blocks of G must consist of two (connected) vertices. Thus G is a tree.

If $G = B + H$, then we have a triangulation tree of G. If this is not the case, then we take another block H' and apply Lemma 2.3 until we have dealt with all blocks of G and thus obtained a triangulation tree of G. Since $B + H$ consists of at least 3 vertices, the exceptional case can appear only for the root vertex.

For the leaves of the tree, it follows from Lemma 2.2 that each one is contained in at least $2^{2n-m'-3}$ labelled outerplanar triangulations, where m' is the number of edges of the corresponding leaf.

For a vertex F of the triangulation tree (F having n vertices and m' edges) which is not exceptional, suppose that its children all have $m' + t$ edges. By induction, each child is contained in at least $2^{2n-(m'+t)-3}$ labelled outerplanar triangulations. Thus, because F has at least 2^t children, F is contained in at least $2^t \cdot 2^{2n-(m'+t)-3} = 2^{2n-m'-3}$ labelled outerplanar triangulations. If F is an exceptional vertex (so it is the root vertex), F has at least 2^{t-1} children. Thus F is contained in at least $2^{t-1} \cdot 2^{2n-(m'+t)-3} = 2^{2n-m'-4}$ labelled outerplanar triangulations.

Thus G is contained in at least 2^{2n-m-3} labelled outerplanar triangulations if the root vertex is not exceptional, and at least 2^{2n-m-4} outerplanar triangulations if it is exceptional. ☐

We conclude with an example which shows that Theorem 2.4 is best possible for $m \approx \frac{4}{3}n$. Observe that the upper bound in Theorem 2.6 on $\left|\mathcal{P}_{n,m}^{\text{out}}\right|$ attains its maximum for $m = \frac{4}{3}n$. We also give two examples which show that Theorem 2.4 can also be very bad for $m \approx n$.

Example. Let n be an even number. Consider the outerplanar graph from Figure 2.2 on $3n$ vertices and $4n-2$ edges.

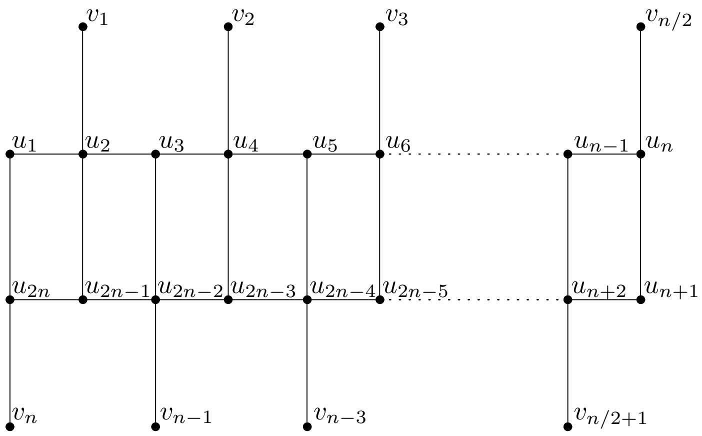

Figure 2.2: Connected outerplanar graph on $3n$ vertices and $4n-2$ edges with 2^{2n-1} labelled outerplanar triangulations.

The number of inner faces is $n-1$. Each inner face can be triangulated in two ways. A vertex v_i, $i = 1, \dots, n-1$, can only be connected to vertex u_{2i-1} or vertex u_{2i+1}, and vertex v_n to u_{2n-1} or u_1. Thus we obtain $2^{n-1} \cdot 2^n = 2^{2n-1}$ outerplanar labelled triangulations. Theorem 2.4 gives a lower bound of $2^{2 \cdot 3n-(4n-2)-3} = 2^{2n-1}$, so Theorem 2.4 is best possible for $m \approx \frac{4}{3}n$.

Example. A circle on n vertices has $\text{Cat}\,(n) = \frac{1}{n-1}\binom{2n-4}{n-2} = 4^{n+o(n)}$ labelled outerplanar triangulations, while Theorem 2.4 (or Lemma 2.2 or Proposition 2.1) gives a lower bound of 2^{n-3}.

Example. A path on n vertices, $n \geq 3$, has $\frac{1}{2}\binom{2n-4}{n-2} = 4^{n+o(n)}$ labelled outerplanar triangulations (see appendix A), while Theorem 2.4 gives a lower bound of $2^{2n-(n-1)-3} = 2^{n-2}$.

2.3 Outerplanar graphs

Theorem 2.5. *Every outerplanar graph $G \in \mathcal{P}_{n,m}^{\text{out}}$ is contained in at least 2^{2n-m-5} labelled outerplanar triangulations.*

Proof. If G is connected, the claim follows immediately from Theorem 2.4.

In the case of two components, adding an arbitrary edge (which preserves outerplanarity) connecting the two components and applying Theorem 2.4 to the resulting graph leads to at least

$$2^{2n-(m+1)-4} = 2^{2n-m-5}$$

outerplanar triangulations.

Consider now the case that G has $k \geq 3$ components.

We are now connecting the components via a circle and triangulate the obtained graph using Theorem 2.4.

Let $v_1, \ldots, v_k$ be fixed vertices with degree at most 2 from $K_1, \ldots, K_k$ ($v_i \in K_i$)[1]. Denote by C_l, $l = 1, \ldots, \frac{1}{2}(k-1)!$, the different labelled cycles which can be formed by $v_1, \ldots v_k$, and by $\sigma_l : [k] \to [k]$ the mapping such that $v_{\sigma_l(i)}$ is the neighbour of v_i at C_l in clockwise order. We now construct G_l by adding the edges $v_i v_{\sigma_l(i)}$, $i \in [k]$, to G. Each G_l has $m+k$ edges and contains the circle C_l (i.e., G_l is not a tree). They are all pairwise incomparable.

Using Theorem 2.4, we can now construct outerplanar triangulations $G_{l,j}$, $l = 1, \ldots, \frac{1}{2}(k-1)!$, $j = 1, \ldots, s$, with $s \geq 2^{2n-(m+k)-3}$. Since $(k-1)! \geq 2^{k-1}$ for $k \geq 3$, this gives at least

$$2^{2n-(m+k)-3} \frac{1}{2}(k-1)! \geq 2^{2n-m-5}$$

outerplanar triangulations. $\square$

2.4 Upper bounds

For $0 < x < 1$ let

$$H(x) := -x \log x - (1-x) \log(1-x)$$

the entropy function, where log denotes the logarithm to base 2 and let $H(0) := H(1) := 0$. It has the property that $\binom{n}{xn} = 2^{H(x)n+o(n)}$ for $0 \leq x \leq 1$.

Theorem 2.6. *For* $0 \leq c \leq 2$,

$$\left|\mathcal{P}^{\mathrm{out}}_{n,cn}\right| \leq n! \cdot 2^{f(c)n+o(n)}, \qquad f(c) := c + 2H\left(\frac{c}{2}\right).$$

The maximum is attained at $c = \frac{4}{3}$, *and*

$$\left|\mathcal{P}^{\mathrm{out}}_{n}\right| \leq n! \cdot 9^{n+o(n)}.$$

Proof. For $G \in \mathcal{P}^{\mathrm{out}}_n$ define by

$$\beta(G) := \left|\left\{H \in \mathcal{T}^{\mathrm{out}}_n \,|G \subseteq H\right\}\right|$$

the number of outerplanar triangulations which contain G. Let

$$\beta(c,n) := \min_{G \in \mathcal{P}^{\mathrm{out}}_{n,cn}} \beta(G).$$

From Theorem 2.5 we know

$$\beta(c,n) \geq 2^{2n-cn-5}.$$

Furthermore, it is known that $|\mathcal{T}^{\mathrm{out}}_n| \leq n! \cdot \mathrm{Cat}(n) \leq n! \cdot 2^{2n}$.

Putting these facts together, we obtain

$$\left|\mathcal{P}^{\mathrm{out}}_{n,cn}\right| \leq \left|\mathcal{T}^{\mathrm{out}}_n\right| \cdot \frac{\binom{2n-3}{cn}}{\beta(c,n)} \leq n! \cdot 2^{2n} \cdot \frac{2^{2H(c/2)n+o(n)}}{2^{(2-c)n-5}} = n! \cdot 2^{f(c)n+o(n)}.$$

Then for $c \in (0,2)$, $\frac{\partial f}{\partial c} = 1 + \log\left(\frac{2-c}{c}\right)$, and $\frac{\partial f}{\partial c}\left(\frac{4}{3}\right) = 0$. One can verify that this maximizes f and $f\left(\frac{4}{3}\right) = \log 9$. Since $\mathcal{P}^{\mathrm{out}}_n = \bigcup_{m=0}^{2n-3} \mathcal{P}^{\mathrm{out}}_{n,m}$, we get

$$\left|\mathcal{P}^{\mathrm{out}}_n\right| = \sum_{m=0}^{2n-3} \left|\mathcal{P}^{\mathrm{out}}_{n,m}\right| \leq (2n-3) \cdot n! \cdot 2^{f\left(\frac{4}{3}\right)n+o(n)} = n! \cdot 2^{(\log 9)n+o(n)} = n! \cdot 9^{n+o(n)}. \qquad \square$$

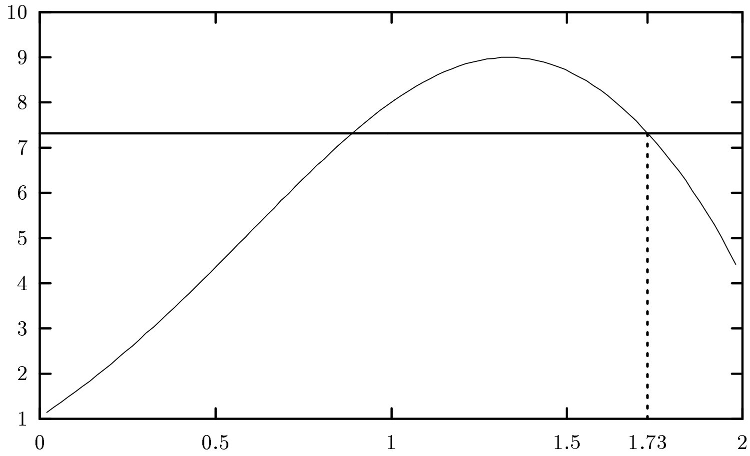

Figure 2.3: The upper bound $2^{f(c)} = \left(\frac{1}{n!}\left|\mathcal{P}^{\mathrm{out}}_{n,cn}\right| - o(n)\right)^{\frac{1}{n}}$ versus c, where $0 < c < 2$, and the asymptotic number 7.32.

Theorem 2.7. *Almost all labelled outerplanar graphs have at most* 1.73 *edges.*

Proof. From [1] we know that

$$\left|\mathcal{P}^{\mathrm{out}}_n\right| > n! \cdot 7.32097^{n+o(n)} = n! \cdot 2^{\log(7.32097)n+o(n)}.$$

Numerical computations give $f(c) < \log(7.32097)$ for $c \geq 1.73$. Then by Theorem 2.6

$$\mathbb{P}\left[G \in \mathcal{P}^{\mathrm{out}}_n \text{ has more than 1.73 edges}\right] = \frac{\left|\bigcup_{c=1.73}^{2} \mathcal{P}^{\mathrm{out}}_{n,cn}\right|}{\left|\mathcal{P}^{\mathrm{out}}_n\right|} \leq \frac{\max\limits_{c\in[1.73,2]} 2^{f(c)n+o(n)}}{2^{\log(7.32097)n+o(n)}} = o(1). \quad \square$$

3 The expected number of vertices of degree 0, 1, and 2

3.1 An upper bound on the expected number of isolated vertices

In the introduction we claimed that $I^{\mathrm{out}}_n \leq \frac{1}{2} + o(1)$. To prove this, we first need some facts about the number of edges which lie on the outer face of an outerplanar graph.

Denote by $t(G)$ the number of edges $\{x, y\} \in E(G)$ for which x and y lie in the same block of $G - \{x, y\}$. Denote by $\kappa_1(G)$ the number of connected components of G, by $\kappa_2(G)$ the number of blocks of G which have size at least 3, and by $f(G)$ the number of faces of G.

Proposition 3.1. *Let $G \in \mathcal{P}^{\mathrm{out}}_n$ be an outerplanar graph with an embedding, where all vertices lie on the outer face, and let $\{x, y\} \in E(G)$. Then x and y lie in the same block of $G - \{x, y\}$ if and only if $\{x, y\}$ lie not on the outer face of G.*

Proof. ($\Leftarrow$) Let $\{x, y\} \in E(G)$ with $\{x, y\}$ not on the outer face of G. Then there exist two inner-disjoint (only first and last vertex in common) paths P_1 and P_2 from x to y in G which do not use $\{x, y\}$. $V(P_1 \cup P_2)$ is biconnected in G. Removing the edge $\{x, y\}$ keeps the biconnectness of $V(P_1 \cup P_2)$, because all its vertices lie on the circle given by P_1 and P_2.

[1]Each connected outerplanar graph has at least one vertex of degree at most 2: For $n \leq 3$, this is trivial. For $n \geq 4$ assume, each vertex has degree at least 3. Then each vertex belongs to the bound of at least 3 faces (no leaves), so $f \geq 3n$. Using $f = 2 - n + m$, we obtain $m \geq 4n - 2$, contradicting $m \leq 2n - 3$.

($\Rightarrow$) If x and y lie in the same block of $G - \{x, y\}$, then there exist two inner-disjoint paths P_1 and P_2 in $G - \{x, y\}$ from x to y. So P_1 and P_2 build a circle whose vertices lie on the outer face of the outerplanar graph $G - \{x, y\}$. Since $|P_1| \geq 3$ and $|P_2| \geq 3$, the edge $\{x, y\}$ can only be embedded in the inner face of that circle, so $\{x, y\}$ cannot lie on the outer face of G. □

Corollary 3.2. *The number of edges on the outer face of $G \in \mathcal{P}_n^{\mathrm{out}}$ is exactly $|E(G)| - t(G)$.*

Lemma 3.3. *For $G \in \mathcal{P}_n^{\mathrm{out}}$,*

$$t(G) = f(G) - \kappa_2(G) - 1.$$

Proof. Let $\{x, y\}$ be an edge of G, where x and y lie in the same block of $G - \{x, y\}$. Then there are two (shortest) inner-disjoint paths P_1 and P_2 from x to y in G which do not use the edge $\{x, y\}$. So $|P_1| > 2$ and $|P_2| > 2$.

The edge $\{x, y\}$ and P_1 form one face of G, $\{x, y\}$ and P_2 form an adjoining one. Removing the edge units these faces, so we get one inner face less.

On the other hand, two adjoining inner faces have at most one edge in common by outerplanarity. The vertices x and y of this edge lie in the same block of $G - \{x, y\}$.

So the edges $\{x, y\}$, where x and y lie in the same block of $G - \{x, y\}$ are those edges which join two inner faces of G. So, how many of these edges do we have?

Denote by f_i, $i = 1, \ldots, \kappa_2(G)$, the number of inner faces of the i'th block of G, where we count only blocks with at least 3 vertices. Then we have $f_i - 1$ edges in the i'th block which belong to two inner faces.

Since all inner faces of G belong to a block of G, the number of edges which belong to two inner faces is exactly

$$t(G) = \sum_{i=1}^{\kappa_2(G)} (f_i - 1) = f(G) - 1 - \kappa_2(G).$$ □

Corollary 3.4. *For $G \in \mathcal{P}_n^{\mathrm{out}}$,*

$$t(G) = |E(G)| - n + \kappa_1(G) - \kappa_2(G).$$

Proof. Denote by f_i, n_i and m_i the number of faces, vertices, and edges, respectively, of the i'th connected component of G, $i = 1, \ldots, \kappa_1(G)$. For each connected component, we use Euler's Formula ($f_i = m_i - n_i + 2$) to obtain

$$f(G) = 1 + \sum_{i=1}^{\kappa_1(G)} (f_i - 1) = 1 + \sum_{i=1}^{\kappa_1(G)} (m_i - n_i + 1) = |E(G_i)| - n + \kappa_1(G) + 1.$$

Thus,

$$t(G) = f(G) - 1 - \kappa_2(G) = |E(G)| - n + \kappa_1(G) + 1 - 1 - \kappa_2(G).$$ □

Theorem 3.5.

$$\left|\mathcal{P}_n^{\mathrm{out}}\right| \geq \left(2n - 1 + \mathbb{E}_{\mathcal{P}_{n-1}^{\mathrm{out}}}[\kappa_2 - \kappa_1]\right) \left|\mathcal{P}_{n-1}^{\mathrm{out}}\right|.$$

Proof. For a graph $G \in \mathcal{P}_{n-1}^{\mathrm{out}}$ we count the number of ways to create a graph of $\mathcal{P}_n^{\mathrm{out}}$ by adding the vertex n. We can

- keep n as an isolated vertex, or
- attach n to one vertex of G, there are $n - 1$ possibilities, or

- attach n to two extremities of an edge which lies on the outer face of G, by Corollary 3.2 there are $|E(G)| - t(G)$ many of them, and using Corollary 3.4 ($t(G) = |E(G)| - (n-1) + \kappa_1(G) - \kappa_2(G)$) we obtain $n - 1 + \kappa_2(G) - \kappa_1(G)$ possibilities.

This gives at least

$$1 + (n-1) + (n - 1 + \kappa_2(G) - \kappa_1(G)) = 2n - 1 + \kappa_2(G) - \kappa_1(G)$$

possibilities to construct a graph of $\mathcal{P}_n^{\text{out}}$ by adding the vertex n to G. So

$$\left|\mathcal{P}_n^{\text{out}}\right| \geq \sum_{G \in \mathcal{P}_{n-1}^{\text{out}}} (2n - 1 + \kappa_2(G) - \kappa_1(G)) = \Big(2n - 1 + \mathbb{E}_{\mathcal{P}_{n-1}^{\text{out}}}[\kappa_2 - \kappa_1]\Big) \cdot \left|\mathcal{P}_{n-1}^{\text{out}}\right|. \qquad \square$$

Corollary 3.6.

$$I_n^{\text{out}} \leq \frac{1}{2} + o(1).$$

Proof. Clearly,

$$I_n^{\text{out}} = n \cdot \mathbb{P}[\text{vertex } n \text{ is isolated}] = n \cdot \frac{\left|\mathcal{P}_{n-1}^{\text{out}}\right|}{\left|\mathcal{P}_n^{\text{out}}\right|},$$

and Theorem 3.5 gives

$$I_n^{\text{out}} \leq n \cdot \frac{\left|\mathcal{P}_{n-1}^{\text{out}}\right|}{\Big(2n - 1 + \mathbb{E}_{\mathcal{P}_{n-1}^{\text{out}}}[\kappa_2 - \kappa_1]\Big)\left|\mathcal{P}_{n-1}^{\text{out}}\right|} \leq \frac{n}{2n - 1 - \mathbb{E}_{\mathcal{P}_{n-1}^{\text{out}}}[\kappa_1]}.$$

From [6] we know that $\mathbb{E}_{\mathcal{P}_{n-1}^{\text{out}}}[\kappa_1] \leq 2$ (see also Introduction), therefore we get

$$I_n^{\text{out}} \leq \frac{n}{2n - 1 - 2} = \frac{1}{2} + o(1). \qquad \square$$

3.2 The expected number of vertices of degree 1 and 2

Theorem 3.7. *Denote by $X_1(G)$ the number of vertices of degree 1 of $G \in \mathcal{P}_n^{\text{out}}$ and by $X_2(G)$ the number of vertices of degree 2 of $G \in \mathcal{P}_n^{\text{out}}$. Then*

$$\begin{aligned}
\mathbb{E}_{\mathcal{P}_n^{\text{out}}}[X_1] &= (n-1)\, I_n^{\text{out}}, \\
\mathbb{E}_{\mathcal{P}_n^{\text{out}}}[X_2] &= 2\left(n - 1 + \frac{1}{n}\mathbb{E}_{\mathcal{P}_{n-1}^{\text{out}}}[\kappa_2 - \kappa_1]\right) \cdot I_n^{\text{out}}.
\end{aligned}$$

If in addition the isolated vertices conjecture is true for outerplanar graphs (and it is, as shown in [1]), then

$$(2n-2)\, I_n^{\text{out}} - o(1) \leq \mathbb{E}_{\mathcal{P}_n^{\text{out}}}[X_2] \leq (2n-1)\, I_n^{\text{out}} + o(1).$$

Proof. Observe that all graphs in $\mathcal{P}_n^{\text{out}}$ in which vertex n has degree exactly one can be obtained from a graph $G' \in \mathcal{P}_{n-1}^{\text{out}}$ on the vertices $[n-1]$ by connecting n to one of the vertices in $[n-1]$, so

$$\mathbb{E}_{\mathcal{P}_n^{\text{out}}}[X_1] = n \cdot \mathbb{P}_{\mathcal{P}_n^{\text{out}}}[\text{vertex } n \text{ has degree } 1] = n \cdot \frac{(n-1)\left|\mathcal{P}_{n-1}^{\text{out}}\right|}{\left|\mathcal{P}_n^{\text{out}}\right|} = (n-1)\, I_n^{\text{out}}.$$

For $\mathbb{E}_{\mathcal{P}_n^{\text{out}}}[X_2]$ observe that we can obtain a graph with vertex n having degree 2 from a graph $G' \in \mathcal{P}_{n-1}^{\text{out}}$ by

1. adding n to both vertices of an edge of G' which lies on the outer face, or

2. adding n to both vertices of an edge $\{x,y\} \notin E(G')$, if these two vertices lie not in the same block of G' (so, adding $\{x,y\}$ to G' preserves outerplanarity).

   Denote the number of these edges by $\mathrm{add}^{\mathrm{out}}(G')$. By Proposition 3.1, $\mathrm{add}^{\mathrm{out}}(G')$ counts the edges $\{x,y\} \notin E(G')$ which lie on the outer face of $G' + \{x,y\}$. So by Corollary 3.2

$$\sum_{G' \in \mathcal{P}_{n-1}^{\mathrm{out}}} \mathrm{add}^{\mathrm{out}}(G') = \sum_{G' \in \mathcal{P}_{n-1}^{\mathrm{out}}} |E(G')| - t(G').$$

Thus, by using Corollary 3.2 and 3.4 we obtain

$$\begin{aligned}
\left|\mathcal{P}_n^{\mathrm{out}}\right| \mathbb{P}_{\mathcal{P}_n^{\mathrm{out}}}[\text{vertex } n \text{ has degree } 2] &= \sum_{G' \in \mathcal{P}_{n-1}^{\mathrm{out}}} \left(|\text{edges on outer face of } G'| + \mathrm{add}^{\mathrm{out}}(G')\right) \\
&= \sum_{G' \in \mathcal{P}_{n-1}^{\mathrm{out}}} \left(|E(G')| - t(G') + |E(G')| - t(G')\right) \\
&= \sum_{G' \in \mathcal{P}_{n-1}^{\mathrm{out}}} \left(2|E(G')| - 2\left(|E(G')| - n + 1 + \kappa_1(G') - \kappa_2(G')\right)\right) \\
&= (2n-2)\left|\mathcal{P}_{n-1}^{\mathrm{out}}\right| + 2 \sum_{G' \in \mathcal{P}_{n-1}^{\mathrm{out}}} \left(\kappa_2(G') - \kappa_1(G')\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_{\mathcal{P}_n^{\mathrm{out}}}[X_2] &= n \cdot \mathbb{P}_{\mathcal{P}_n^{\mathrm{out}}}[\text{vertex } n \text{ has degree } 2] \\
&= n \cdot \frac{(2n-2)\left|\mathcal{P}_{n-1}^{\mathrm{out}}\right| + 2\sum_{G' \in \mathcal{P}_{n-1}^{\mathrm{out}}} \left(\kappa_2(G') - \kappa_1(G')\right)}{\left|\mathcal{P}_n^{\mathrm{out}}\right|} \\
&= 2n \cdot (n-1) \cdot \frac{\left|\mathcal{P}_{n-1}^{\mathrm{out}}\right|}{\left|\mathcal{P}_n^{\mathrm{out}}\right|} + 2\mathbb{E}_{\mathcal{P}_{n-1}^{\mathrm{out}}}[\kappa_2 - \kappa_1] \cdot \frac{\left|\mathcal{P}_{n-1}^{\mathrm{out}}\right|}{\left|\mathcal{P}_n^{\mathrm{out}}\right|} \\
&= 2(n-1) \cdot I_n^{\mathrm{out}} + 2\mathbb{E}_{\mathcal{P}_{n-1}^{\mathrm{out}}}[\kappa_2 - \kappa_1] \cdot \frac{I_n^{\mathrm{out}}}{n}.
\end{aligned}$$

For the asymptotic behavior recall that $\mathbb{E}_{\mathcal{P}_{n-1}^{\mathrm{out}}}[\kappa_1] \leq 2$. Furthermore, because two blocks which have at least 3 vertices both can only overlap in at most one vertex, we have $\kappa_2 \leq \frac{n-1}{2}$. Therefore we obtain

$$\mathbb{E}_{\mathcal{P}_n^{\mathrm{out}}}[X_2] \sim 2\left(n - 1 + \frac{1}{n}\mathbb{E}_{\mathcal{P}_{n-1}^{\mathrm{out}}}[\kappa_2]\right) I_n^{\mathrm{out}} \leq 2\left(n - 1 + \frac{n-1}{2n}\right) I_n^{\mathrm{out}} \sim 2\left(n - \frac{1}{2}\right) I_n^{\mathrm{out}}. \qquad \square$$

A Counting the number of outerplanar triangulations of a path

Denote by $p(n)$ the number of outerplanar triangulations of a path P on n labelled vertices $3, \ldots, n$. For the sake of simplicity we define $p(n) := 0$ for $n \leq 1$ and $p(2) := 1$. Recall from Proposition 2.1 that the n-th Catalan-number $\mathrm{Cat}(n) = \frac{1}{n-1}\binom{2n-4}{n-2}$ gives the number of triangulations of the interior of a circle on n vertices with a fixed vertex-labelling.

To compute $p(n)$, we add an edge from the vertex n to another vertex to obtain a graph, consisting of a circle and a smaller path. Then we can triangulate the circle separately and apply the same procedure to the smaller path. This leads to a recursion formula for $p(n)$, which we solve using generating functions. Adding the edge, i.e., constructing the cycle, fixes a part of the Hamiltonian cycle which we use to classify the triangulations.

The following propositions should be clear:

Proposition A.1. *Let C be the unique Hamiltonian cycle of an outerplanar triangulation of P. Then the edges $\{1,2\}$ and $\{n-1,n\}$ belong to C.*

For an outerplanar triangulation of P, denote by s the neighbour of n on the Hamiltonian cycle C which satisfies $s < n-1$ (for $n \geq 3$).

Proposition A.2. *Let C be the unique Hamiltonian cycle of an outerplanar triangulation of P. If $s > 1$ and $n > 3$, then $\{s, s+1\}$ is not an edge of the Hamiltonian cycle. If $s = 1$ or $n = 3$, then $C = P + \{n, 1\}$, so we have $\mathrm{Cat}(n)$ outerplanar triangulations.*

Lemma A.3. *The number $p(n)$ of outerplanar triangulations of a path P on n labelled vertices, $n \geq 3$, satisfies the formula*

$$p(n) = \sum_{s=2}^{n-1} \mathrm{Cat}(n-s+2)\, p(s).$$

Proof. Clearly, each triangulation of P contains a circle $s, s+1, \ldots, n$ for a vertex $s \in \{1, \ldots, n-2\}$. We separate the triangulations T of P by those that contain the circle $s, s+1, \ldots, n$ such that

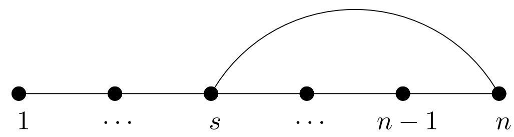

Figure A.1: Separating the triangulations of P by fixing the edge $\{s, n\}$.

$s \ldots n$ is part of the Hamiltonian cycle of T. Using Propositions A.1 and A.2 shows that we can then triangulate the circle $s, s+1, \ldots, n$ and the path $1, \ldots, s, s+1$ separately, i.e., (using $p(2) = 1$)

$$p(n) = \underbrace{\mathrm{Cat}(n)}_{s=1} + \sum_{s=2}^{n-2} \mathrm{Cat}(n-s+1)\, p(s+1) = \sum_{s=1}^{n-2} \mathrm{Cat}(n-s+1)\, p(s+1). \qquad \square$$

Theorem A.4. *The number of outerplanar triangulations of a path P on n labelled vertices, $n \geq 3$, is exactly*

$$p(n) = \frac{1}{2}\binom{2n-4}{n-2}.$$

Proof. Let us define $\mathrm{Cat}(n) := 0$ for $n \leq 2$, so we can write

$$p(n) = \sum_{s=2}^{\infty} \mathrm{Cat}(n-s+2)\, p(s)$$

for $n \geq 3$. To solve the recursion from Lemma A.3 we define two generating functions: Let $P(x)$ be the ordinary generating function of $\{p(n)\}_{n=0}^{\infty}$ and let $C(x)$ be the ordinary generating function of $\{\mathrm{Cat}(n)\}_{n=0}^{\infty}$. Then

$$\begin{aligned}
P(x) &= \sum_{n=0}^{\infty} p(n)\, x^n = x^2 + \sum_{n=3}^{\infty}\sum_{s=2}^{\infty} \mathrm{Cat}(n-s+2)\, p(s)\, x^n \\
&= x^2 + \sum_{n=3}^{\infty} \mathrm{Cat}(n)\, p(2)\, x^n + \sum_{s=3}^{\infty} p(s)\, x^{s-2} \sum_{n=3}^{\infty} \mathrm{Cat}(n-s+2)\, x^{n-s+2} \\
&= x^2 + C(x) + \sum_{s=3}^{\infty} p(s)\, x^{s-2} C(x) = x^2 + C(x) + \frac{1}{x^2}\left(P(x) - x^2\right) C(x) \\
&= x^2 + \frac{1}{x^2} C(x)\, P(x),
\end{aligned}$$

so

$$P(x) = \frac{x^4}{x^2 - C(x)}.$$

Now let $a(n) = \frac{1}{n+1}\binom{2n}{n}$, $n \geq 0$, so $c(n) = a(n-2)$ for $n \geq 3$, but $a(0) = 1$. Then the ordinary generating function $A(x) := \sum_{n=0}^{\infty} a(n) x^n$ of $\{a(n)\}_{n=0}^{\infty}$ satisfies

$$C(x) = x^2 (A(x) - 1).$$

From [8] we know that $A(x) = \frac{1-\sqrt{1-4x}}{2x}$, so

$$C(x) = \frac{1}{2} x \left(1 - \sqrt{1-4x} - 2x\right)$$

and

$$P(x) = \frac{x^4}{x^2 - \frac{1}{2} x \left(1 - \sqrt{1-4x} - 2x\right)}.$$

This can be simplified to

$$P(x) = \frac{2x^3}{\sqrt{1-4x}\left(1 - \sqrt{1-4x}\right)} = \frac{2x^3\left(1 + \sqrt{1-4x}\right)}{\sqrt{1-4x} \cdot 4x} = \frac{1}{2} x^2 \cdot \frac{1}{\sqrt{1-4x}} + \frac{1}{2} x^2$$

and from [9, Section 2.5] we obtain $\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$, thus we get

$$P(x) = \sum_{n=0}^{\infty} \frac{1}{2} \binom{2n}{n} x^{n+2} + \frac{1}{2} x^2$$

and finally

$$p(n) = \frac{1}{2} \binom{2(n-2)}{n-2} x^n$$

for $n \geq 3$. □

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