Asymptotic Enumeration of Unlabelled Outerplanar Graphs

Diploma Thesis

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Abstract

Counting labelled planar graphs and labelled or unlabelled planar maps has been a well-studied problem for many years. The growth rates for labelled planar, labelled series-parallel, and labelled outerplanar graphs have been found recently [3, 19]. In this thesis we investigate unlabelled outerplanar graphs. The analysis of the two-connected components of an unlabelled outerplanar graph is essential for the construction of generating functions for the connected and general case. Using singularity analysis, we show that the number g_n of unlabelled outerplanar graphs on n vertices is asymptotically $g\,n^{-5/2}\rho^{-n}$ where g and ρ can be approximated, $g \approx 0.00909941$, $\rho^{-1} \approx 7.50360$, and the number of connected outerplanar graphs on n vertices is asymptotically $c\,n^{-5/2}\rho^{-n}$ where c can be approximated, $c \approx 0.00760471$, and ρ as before. Therefore, the asymptotic probability of a random unlabelled outerplanar graph (sampled uniformly at random) to be connected can be approximated by $\frac{c}{g} \approx 0.845721$. Furthermore, we show that the expected number of components in a random unlabelled outerplanar graph is asymptotically ≈ 1.17847 and the expected number of isolated vertices is asymptotically ≈ 0.153761. Moreover, we approximate the growth rate for bipartite unlabelled outerplanar graphs and show that the chromatic number of a random unlabelled outerplanar graph is three with probability tending to one as $n \to \infty$. Finally, we investigate the number of edges in a random unlabelled outerplanar graph on n vertices and show that their limit distribution is Gaussian with mean $\approx 1.54894n$.

Contents

Chapter 1

Introduction

Counting graphs of different types has been an interesting task for many years. While the question for the number of simple graphs on a fixed number of labelled vertices has a simple answer, it already becomes much more challenging when the unlabelled case is considered [20, Ch. 4]. Similarly, the number of trees on n labelled vertices has already been found by Cayley in 1897 to be exactly n^{n-2} [10], while Otter was the first one who handled the unlabelled case [27]. Instead of closed formulas, only recursions are available and it is still unknown whether the growth constant (the quotient between the number of trees on $n+1$ and that on n unlabelled vertices as $n \to \infty$) is algebraic or transcendent. However, the asymptotic number of unlabelled trees can be approximated [15, 27].

While trees belong to the simplest kind of connected graphs, one might turn the attention to more complicated classes of graphs. A class of graphs which seems to be still tractable for enumeration is the class of labelled planar graphs. Three-connected planar graphs have the property of possessing a unique embedding in the sphere [36], so their counting formula is related to the one of three-connected planar maps, i.e., embeddings of three-connected planar graphs [26]. Bender, Gao, and Wormald made use of this property to derive the counting formulas for labelled two-connected planar graphs [2] and very recently this work has been extended by Giménez and Noy [19] to the enumeration of labelled planar graphs. The growth constant for labelled planar graphs can be computed with arbitrary precision and is ≈ 27.2269. Many interesting properties of a labelled planar graph which is drawn uniformly at random from the set of all planar graphs, such as the probability that a fixed connected planar graph appears linearly many times, have been studied by McDiarmid, Steger, and Welsh in [25]. It is also known how to generate rooted three-connected planar graphs uniformly at random [4, 30]. Denise, Vasconcellos, and Welsh specified a Markov chain on the space of all labelled planar graphs, whose limit distribution is the uniform distribution over all labelled planar graphs [12], but it is still unclear whether this chain is rapidly mixing.

For unlabelled planar graphs, no closed counting formula is known for the general case. However, recursive counting formulas for rooted two-connected planar graphs and rooted cubic planar graphs have been derived by Bordirsky, Gröpl, and Kang [5, 6]. On the other hand, planar maps have been studied intensively in the past. It starts with the work of Tutte [33] on the enumeration of rooted oriented planar maps which have the property that only the trivial automorphism fixes the root. Statistical properties of such a map, drawn uniformly at random, have been studied by Banderier et. al. [1]. Also the enumeration of unrooted planar maps has been a well-studied problem, see, e.g., the works of Liskovets, Walsh, and Fusy [17, 22, 23].

A recent overview about growth constants for several types of planar graphs has been given by Finch [14]. One such subclass is the set of outerplanar graphs, which can be characterized in terms of forbidden minors (namely $K_{2,3}$ and K_4) [11]. They have the property that their two-connected components can be identified with dissections of a polygon [36]. Bodirsky and Kang have derived recursion formulas for (labelled and rooted unlabelled) outerplanar graphs which can be used to generate such graphs uniformly at random in polynomial time [7]. Furthermore, Bodirsky, Giménez, Kang, and Noy computed the growth constant for labelled outerplanar graphs and labelled series-parallel graphs (K_4-minor-free graphs) [3], where the growth constant for the former has been estimated to be ≈ 7.32098, and the latter to be ≈ 9.32883. Labelled bipartite outerplanar graphs were counted by Löffler [24], the growth constant for the two-connected case is ≈ 2.96956, for the connected case and for the not necessarily connected case it is ≈ 4.40364.

Both the enumeration of labelled graphs and that of unlabelled graphs have the common property that once the enumeration of the two-connected components in terms of counting formulas is well understood, the extension to counting formulas for the connected or general case does not appear to be much difficult. For labelled outerplanar graphs, the two-connected components can be counted by the well-known bracketing numbers [31, A001003]. For the unlabelled case, Read derived the generating functions for the number of dissections of a convex polygon [29]. In this work, we will extend the work of Read to the enumeration of unlabelled outerplanar graphs.

When counting unlabelled structures, one has to deal with automorphism group. In other words, each unlabelled graph has a group of adjacency-preserving permutations of vertices assigned to it. Understanding and enumerating these automorphism groups turn out to be the essential element for the enumeration of unlabelled graphs. The concept of the cycle index of an automorphism group will be very helpful in this context, and the classical enumeration theorem of Pólya then builds a bridge to the enumeration of unlabelled structures [20, Ch. 2]. Enumeration of structures is often based on decomposing them into simpler structures. For labelled graphs, where all vertices are fixed, it is not difficult to find an anchor point in the graph where the decomposition is performed, but in the unlabelled case, this is less trivial. One method is to fix one vertex and to decompose this so-called rooted unlabelled graph, using the fixed vertex as an anchor point. Afterwards, one has to think how to unroot the rooted graphs.

When the enumerating functions are known (explicitly or implicitly), one is interested in closed formulas and asymptotics for their coefficients. Closed exact formulas can sometimes be derived with the Lagrange inversion formula [20, Sec. 1.7], but this seems to be only successful for simple structures. On the other hand, asymptotics are often easier to obtain and sometimes give surprisingly good estimates on the exact numbers. The asymptotic numbers are mainly determined by the dominant singularity of the generating function. Expanding this function around its dominant singularity allows the computation of growth constants and subexponential factors with almost arbitrary precision (see [16]).

1.1 Main theorems and outline

We show how to derive counting formulas (cycle indices and generating functions) for rooted and unrooted unlabelled dissections and how they can be used to derive the corresponding functions for rooted and unrooted unlabelled connected outerplanar graphs. Exact numbers can be computed, but the growth rate and subexponential factors can be only approximated, as for unlabelled trees. The same holds for unlabelled outerplanar graphs, we can compute some exact numbers, but only approximate the asymptotics. Moreover, we can show some limit laws regarding the number and type of the components, the chromatic number, and the number of edges of a random unlabelled outerplanar graph. The main results of this thesis are as follows:

Theorem (asymptotic estimates for outerplanar graphs). *The number of unlabelled two-connected outerplanar graphs d_n, unlabelled connected outerplanar graphs c_n, and unlabelled outerplanar graphs g_n, on n vertices have the asymptotic estimates*

$$\begin{aligned} d_n &\sim d\,n^{-\frac{5}{2}}\delta^{-n}, \\ c_n &\sim c\,n^{-\frac{5}{2}}\rho^{-n}, \\ g_n &\sim g\,n^{-\frac{5}{2}}\rho^{-n}, \end{aligned}$$

with growth-rates $\delta^{-1} = 3 + 2\sqrt{2} \approx 5.82843$ *and* $\rho^{-1} \approx 7.50360$*, and constants* $d \approx 0.00596026$*,* $c \approx 0.00760471$*, and* $g \approx 0.00909941$*. (see Corollary 2.15, Theorem 3.5, Theorem 3.8 and Section 3.1.3.)*

Theorem (typical properties of a random outerplanar graph)**.** *The following statements about connectivity, the number of components, and the number of isolated vertices hold.*

1. *The probability for a random outerplanar graph to be connected is asymptotically* $\frac{c}{g} \approx 0.845721$*. (see Theorem 3.9)*
2. *The expected number of components in a random outerplanar graph is asymptotically* ≈ 1.17847*. (see Theorem 3.10)*
3. *The distribution of the number of isolated vertices in a random outerplanar graph is asymptotically geometric with parameter* ρ*. In particular, the expected number of isolated vertices in a random outerplanar graphs is asymptotically* $\frac{\rho}{1-\rho} \approx 0.153761$*. (see Corollary 3.12)*

Theorem (asymptotic estimates for bipartite outerplanar graphs)**.** *The number of bipartite unlabelled two-connected outerplanar graphs* $(d_{\mathrm{b}})_n$*, unlabelled connected outerplanar graphs* $(c_{\mathrm{b}})_n$*, and unlabelled outerplanar graphs* $(g_{\mathrm{b}})_n$ *on* n *vertices have the asymptotic estimates*

$$\begin{aligned}(d_{\mathrm{b}})_n &\sim \begin{cases} d_{\mathrm{b}}(n)\, \delta_{\mathrm{b}}^{-n}, & n \text{ even},\\ 0, & n \text{ odd},\end{cases}\\ (c_{\mathrm{b}})_n &\sim c_{\mathrm{b}}\, n^{-\frac{5}{2}}\, \rho_{\mathrm{b}}^{-n},\\ (g_{\mathrm{b}})_n &\sim g_{\mathrm{b}}\, n^{-\frac{5}{2}}\, \rho_{\mathrm{b}}^{-n},\end{aligned}$$

with growth-rates $\delta_{\mathrm{b}}^{-1} = \left(\sqrt{\frac{1}{8}\left(71 - 17\sqrt{17}\right)}\right)^{-1} \approx 2.96956$ *and* $\rho_{\mathrm{b}}^{-1} \approx 4.57717$*, subexponential factor* $d_{\mathrm{b}}(n)$ *satisfying* $\limsup_{n\to\infty} \left(d_{\mathrm{b}}(n)\right)^{1/n} = 1$*, and constants* c_{b} *and* g_{b} *which can be approximated. (see Theorems 3.26, 3.29, and 3.31)*

Theorem (chromatic number)**.** *The chromatic number of a random outerplanar graph is three with probability tending to one as* $n \to \infty$*. (see Theorem 3.32)*

Theorem (number of edges)**.** *The limit distribution of the number of edges in a random two-connected outerplanar graph on* n *vertices is Gaussian with expectation* $\left(1 + \frac{1}{\sqrt{2}}\right) n \approx 1.70711n$ *and variance* $\frac{1}{4\sqrt{2}} n \approx 0.176777n$*. (see Theorem 3.47)*

The distribution of the number of edges in a random outerplanar graph on n *vertices is asymptotically Gaussian with expectation* $\approx 1.54894n$ *and variance* $\approx 0.227504n$*. (see Theorem 3.48)*

Furthermore, we give exact numbers for two-connected unlabelled outerplanar graphs (Table 2.3), connected unlabelled outerplanar graphs (Table 3.1), and unlabelled outerplanar graphs (Table 3.4) with up to 25 vertices. Similarly, we have exact numbers for bipartite two-connected unlabelled outerplanar graphs with up to 20 vertices (Table 3.5) and for bipartite connected unlabelled outerplanar graphs and bipartite unlabelled outerplanar graphs with up 12 vertices (Table 3.6). Finally, we count the number of graphs with inclusion of the number of edges, and hence we have exact numbers for two-connected outerplanar graphs (Tables 3.7, 3.8, and 3.8) with up to 15 vertices and up to 27 edges, and for connected and not necessarily connected outerplanar graphs with up to 10 vertices and 17 edges (Tables 3.10, 3.11, and 3.12).

The rest of this chapter is devoted to an introduction into the theory of counting unlabelled structures and singularity analysis. First, we establish the necessary notation and theorems for the enumeration of rooted and unrooted unlabelled graphs, following the standard reference [20].

Then we show how the constructed generating functions can be used to obtain asymptotic estimates of their coefficients. We will follow the book of Flajolet and Sedgewick [16].

In the second chapter we investigate dissections. Following Read [29] we obtain the cycle index and generating functions for rooted and unrooted dissections. Also asymptotics are easily obtained there.

The cycle indices and generating functions for dissections are then used in the third chapter to derive the corresponding functions for rooted and unrooted unlabelled connected outerplanar graphs. Furthermore we prove the above stated theorems about the asymptotic growth constants and limit laws for outerplanar graphs.

We conclude this thesis with a review of the achieved results and an outlook on possible or desirable extensions in Chapter four.

1.2 Counting unlabelled structures

Counting unlabelled graphs has two aspects. First, one has to enumerate the different structures, i.e., without consideration of possible automorphisms, and secondly the enumeration of the permutations that can occur. The former requires often some fixed point(s) so that symmetry is destroyed and a decomposition method can be applied. The decomposition then yields a functional equation of the generating function we are seeking for. For the latter, we reformulate the counting problem, using some group-theoretic terminology, such that the answer can be obtained by finding the number of orbits in a permutation group. Then, Burnside's lemma, the concept of the cycle index and Pólya's theorem allow us to obtain the ordinary generating function of the rooted unlabelled structure we are investigating. Furthermore, we can show how to construct the cycle indices of a class of graphs from the cycle indices of their two-connected components by composition. Finally, the concept of the cycle index of the permutation group of an unlabelled graph turns out to be equivalently well suited to switch between rooted and unrooted graphs as the exponential generating function for labelled graphs. The material for this section is taken from [20], except for Theorem 1.18 which is original work.

Let $X = \{1, \ldots, n\}$ be a finite set and A be a group of permutations on the object set X. We call $|A|$ the *order* of A and $|X| = n$ the *degree* of A. Recall that each permutation $\alpha \in A$ can be uniquely written as a product of disjoint cycles.

Definition 1.1 (cycle index). Let $\alpha \in A$ and for $k = 1, \ldots, n$, let $j_k(\alpha)$ be the number of cycles of length k in the disjoint cycle decomposition of α. Then the *cycle index* of A is defined as the following polynomial in the formal variables $s_1, \ldots, s_n$:

$$Z(A) := Z(A; s_1, \ldots, s_n) := \frac{1}{|A|} \sum_{\alpha \in A} \prod_{k=1}^{n} s_k^{j_k(\alpha)}.$$

Example. Let $S_3 = \{(1)(2)(3), (1)(23), (2)(13), (3)(12), (123), (132)\}$ be the symmetry group of degree 3. Then $Z(S_3) = \frac{1}{6}\left(s_1^3 + 3s_1 s_2 + 2s_3\right)$.

The following groups will play an important role in the next chapters:

Example. The *identity group* E_n of order n has cycle index $Z(E_n) = s_1^n$.

Let S_n be the *symmetric group* of degree n. It has the cycle index

$$Z(S_n) = \frac{1}{n!} \sum_{(j) \vdash n} h(j) \prod_{k=1}^{n} s_k^{j_k}, \qquad h(j) = \frac{n!}{\prod_k k^{j_k} j_k!},$$

where the sum is over all partitions $(j) = (j_1, \ldots, j_n)$ of n, j_k is the number of parts equal to k, and $h(j)$ is the number of permutations in S_n whose cycle decomposition determines the partition (j). For simplicity, we let $Z(S_0) := 1$. It is known that

$$\sum_{n \geq 0} Z(S_n) = \exp\left(\sum_{k \geq 1} \frac{1}{k} s_k\right) \tag{1.1}$$

and

$$Z(S_n) = \frac{1}{n} \sum_{k=1}^{n} s_k Z(S_{n-k}). \tag{1.2}$$

Let C_n be the *cyclic group*, generated by the cycle $(1\,2\,3\,\cdots\,n)$. Then

$$Z(C_n) = \frac{1}{n} \sum_{k|n} \varphi(k)\, s_k^{n/k}, \tag{1.3}$$

where φ is the so-called Euler-function, defined as follows: $\varphi(n) := n \prod_{p|n} \left(1 - p^{-1}\right)$, $n \in \mathbb{N}$, where the product is over all prime numbers p which divide n.

Let D_n be the *dihedral group*, generated by the reflection $(1\,n)\,(2\,(n-1))\,(3\,(n-2))\cdots$ and the cycle $(1\,2\,3\,\cdots\,n)$. Then

$$Z(D_n) = \frac{1}{2} Z(C_n) + \begin{cases} \frac{1}{2} s_1 s_2^{(n-1)/2}, & n \text{ odd}, \\ \frac{1}{4}\left(s_2^{n/2} + s_1^2 s_2^{(n-2)/2}\right), & n \text{ even}. \end{cases} \tag{1.4}$$

1.2.1 Burnside's lemmas

Definition 1.2 (similar, orbit)**.** Two objects $x, y \in X$ are called *similar* or *A-equivalent*, if there exists a permutation $\alpha \in A$ such that $\alpha x = y$. This defines an equivalence relation on X. Its equivalence classes are called *orbits*.

Definition 1.3 (stabilizer)**.** The *stabilizer* of $x \in X$ are those permutations in A which fix x, $A(x) := \{\alpha \in A \,|\, \alpha(x) = x\}$.

The stabilizer of an element $x \in X$ determines a subgroup of A. Observe that if $x, y \in X$ share the same orbit, then $|A(x)| = |A(y)|$. The following lemma on the relation between $|A(y)|$ and $|A|$ will be helpful.

Lemma 1.4 ([20, page 38])**.** *Let $y \in X$ have orbit $Y \subseteq X$. Then $|A| = |A(y)|\,|Y|$.*

Proof. We write $A = \bigcup_{i=1}^{m} \alpha_i A(y)$ for some m and some α_i, $i = 1, \ldots, m$, such that $\alpha_i A(y) \cap \alpha_j A(y) = \emptyset$, $i \neq j$. This allows us to identify $\alpha_i A(y)$ with $\alpha_i(y) \in Y$: For $i \neq j$, $\alpha_i(y) \neq \alpha_j(y)$, since otherwise $\alpha_j^{-1}\alpha_i \in A(y)$. Hence $\alpha_i \in \alpha_j A(y) = \left\{\gamma \,\middle|\, \left(\alpha_j^{-1}\gamma\right)(y) = y\right\}$, and thus $\alpha_i A(y) \cap \alpha_j A(y) \neq \emptyset$. Therefore, the relationship $\alpha_i A(y) \leftrightarrow \alpha_i(y)$ is one to one.

For each $y' \in Y$ there exists an α such that $\alpha(y) = y'$. From the above partition of A, there exists an $\gamma \in A(y)$ and an $i^* \in \{1, \ldots, m\}$ such that $\alpha = \alpha_{i^*}\gamma$. Then $y' = \alpha_{i^*}(y)$, and thus every $y' \in Y$ corresponds to a coset $\alpha_i A(y)$. On the other hand, to each coset $\alpha_i A(y)$ corresponds $y' := \alpha_i(y) \in Y$. Hence, $m = |Y|$. □

Notation. We denote the *number of orbits* of A by $N(A)$.

Lemma 1.5 (Burnside's lemma [20, page 39])**.**

$$N(A) = \frac{1}{|A|} \sum_{\alpha \in A} j_1(\alpha).$$

Proof. Let $m = N(A)$ and let $X_1, \ldots, X_m$ be the orbits of X. Let $x_i \in X_i$, $i = 1, \ldots, m$. From Lemma 1.4 we know $|A| = |A(x_i)| |X_i|$, and thus

$$N(A)|A| = \sum_{i=1}^{m} |A(x_i)| |X_i|.$$

For $x \in X_i$ it holds $|A(x)| = |A(x_i)|$. Hence, we obtain

$$N(A)|A| = \sum_{x \in X} |A(x)| = \sum_{x \in X} \sum_{\alpha \in A(x)} 1 = \sum_{\alpha \in A} \sum_{\substack{x \in X: \\ x = \alpha x}} 1 = \sum_{\alpha \in A} j_1(\alpha). \qquad \square$$

Notation. For a union $Y \subseteq X$ of orbits of A, let $A|Y$ be the permutations on Y obtained by restricting those of A onto Y. Furthermore, let $j_1(\alpha|Y) = |\{y \in Y \,|\alpha(y) = y\}|$ be the number of elements in Y which are fixed by α.

Lemma 1.6 (restricted form of Burnside's lemma [20, page 40])**.**

$$N(A|Y) = \frac{1}{|A|} \sum_{\alpha \in A} j_1(\alpha|Y).$$

Proof. Analog to the proof of Lemma 1.5. $\square$

Lemma 1.7 (weighted form of Burnside's lemma [20, page 40])**.** *For a weight function $w : X \to \mathbb{Z}$ which is constant on the orbits of A,*

$$\sum_{i=1}^{m} w(X_i) = \frac{1}{|A|} \sum_{\alpha \in A} \sum_{\substack{x \in X: \\ x = \alpha x}} w(x)$$

where $w(Y) := w(y)$ for some $y \in Y$, Y an orbit of A.

Proof.

$$|A| \sum_{i=1}^{m} w(X_i) = \sum_{i=1}^{m} |A(x_i)| |X_i| w(X_i) = \sum_{x \in X} |A(x)| w(x) = \sum_{x \in X} \sum_{\substack{x \in X: \\ x = \alpha x}} w(x). \qquad \square$$

1.2.2 Pólya's theorem

Next we show how Burnside's lemma can be used to derive the generating function for configurations that are "compositions" of "figures" with given generating function, where the composition is guided by a permutation group. We start with the definition of "composition" more formally.

Definition 1.8 (power group)**.** For two permutation groups A on X and B on Y the *power group* B^A of permutations on $Y^X := \{f : X \to Y\}$ is the set of pairs $(\alpha, \beta) \in A \times B$ with component-wise concatenation, $(\alpha, \beta)(\alpha', \beta') := (\alpha\alpha', \beta\beta')$ for $(\alpha, \beta), (\alpha', \beta') \in A \times B$. The elements of B^A act on Y^X by $((\alpha, \beta)(f))(x) := \beta(f(\alpha(x))) = \beta f(\alpha x)$ for $f \in Y^X$ and $x \in X$.

Now let $B = E = \{\varepsilon\}$ be the identity group, so $((\alpha, \varepsilon)(f))(x) = f(\alpha x)$, $\alpha \in A$, $f \in Y^X$, $x \in X$.

For a weight-function $w : Y \to \mathbb{N} = \{0, 1, 2, \ldots\}$ with $\left|w^{-1}(k)\right| < \infty$, $k \in \mathbb{N}$, let $c_k := \left|w^{-1}(k)\right|$ be the number of *figures* in Y of weight k. Then define by $c(x) = \sum_k c_k x^k$ the *figure counting series.*

For $f \in Y^X$ we let $w(f) := \sum_{x \in X} w(f(x))$. Observe that $w(f) = w(g)$ if f and g share the same orbit of E^A, i.e., there exists an $\alpha \in A$ such that $(\alpha, \varepsilon) f = g$. Hence, w is constant on the orbits of E^A and we can define $w(F) := w(f)$ for some $f \in F$, F an orbit of E^A. Let further C_k be the number of orbits of E^A with weight k. Then define by $C(x) = \sum_k C_k x^k$ the *function* or *configuration counting series.*

We will make excessive use of the following from now on.

Theorem 1.9 (Pólya [20, page 42])**.**

$$C(x) = Z(A; c(x)) := Z\left(A; c(x), c\left(x^2\right), c\left(x^3\right), \ldots\right)$$

where $Z(A; c(x))$ is obtained from $Z(A)$ in Definition 1.1 by replacing s_k by $c\left(x^k\right)$, $k \geq 1$.

Proof. For $k \in \mathbb{N}$ and $\alpha \in A$ we let

$$\psi(\alpha, k) := \left|\left\{f \in Y^X \,|w(f) = k,\ (\alpha, \varepsilon) f = f\right\}\right|$$

be the number of functions of weight k which are fixed by α. When we apply the restricted form of Burnside's Lemma, Lemma 1.6, to E^A with the restriction on functions of size k, we obtain

$$\begin{aligned} C_k = N\left(E^A \left|\left\{f \in Y^X \,|w(f) = k\right\}\right.\right) &= \frac{1}{|E^A|} \sum_{(\alpha;\varepsilon) \in E^A} j_1\left((\alpha, \varepsilon) \left|\left\{f \in Y^X \,|w(f) = k\right\}\right.\right) \\ &= \frac{1}{|A|} \sum_{\alpha \in A} \left|\left\{f \in Y^X \,|(\alpha, \varepsilon)(f) = f, w(f) = k\right\}\right| \\ &= \frac{1}{|A|} \sum_{\alpha \in A} \psi(\alpha, k). \end{aligned}$$

Thus

$$C(x) = \frac{1}{|A|} \sum_{k=0}^{\infty} \sum_{\alpha \in A} \psi(\alpha, k) x^k = \frac{1}{|A|} \sum_{\alpha \in A} \sum_{k=0}^{\infty} \psi(\alpha, k) x^k,$$

where $\sum_k \psi(\alpha, k) x^k$ is the counting series for all functions which are fixed by (α, ε). We are now seeking for another representation of $\sum_k \psi(\alpha, k) x^k$.

Let $f \in Y^X$ be a function which is fixed by α and let z_r be a cycle of length r in α. Since f is fixed by α, f has to be constant on the elements of z_r. If $f(x) = y$ for $x \in z_r$ and $w(y) = k$, then the cycle z_r contributes rk to the weight $w(f)$. In Y there are c_k elements with weight k as choice for $f(x)$, so we have c_k possibilities to define f for the elements of z_r such that f is fixed by (α, ε) and contributes rk to $w(f)$. Hence, $c(x^r) = \sum c_k x^{rk}$ enumerates the number of possibilities to define the elements of z_r such that f is fixed by (α, ε). Next, $c(x^r)^{j_r(\alpha)}$ enumerates (by weight) the options to define functions fixed by α on all cycles of length r in α and $\prod_{r=1}^{n} c(x^r)^{j_r(\alpha)}$ enumerates the possibilities to define functions fixed by α. Therefore,

$$\sum_{k=0}^{\infty} \psi(\alpha, k) x^k = \prod_{r=1}^{n} c(x^r)^{j_r(\alpha)}$$

and finally

$$C(x) = \frac{1}{|A|} \sum_{\alpha \in A} \prod_{k=1}^{n} c\left(x^k\right)^{j_k(\alpha)} = Z\left(A; c(x), c\left(x^2\right), \ldots\right). \qquad \square$$

This theorem can be generalized to the case of more than one counting variable:

Theorem 1.10 (Pólya [20, page 45])**.** *For a weight function $w : Y \to \mathbb{N}^n$ let $c(x_1, \ldots, x_n)$ be the figure counting series with coefficient $\left|w^{-1}(r_1, \ldots, r_n)\right| < \infty$ of $x_1^{r_1} \cdots x_n^{r_n}$. Let $C(x_1, \ldots, x_n)$ be the corresponding function counting series, which has the number of orbits of weight $(t_1, \ldots, t_n)$ as coefficient of $x_1^{t_1} \cdots x_n^{t_n}$. Let $Z(A; c(x_1, \ldots, x_n))$ be the polynomial obtained by replacing each variable s_k in $Z(A)$ by $c\left(x_1^k, \ldots, x_n^k\right)$. Then*

$$C(x_1, \ldots, x_n) = Z(A; c(x_1, \ldots, x_n)).$$

We illustrate Pólya's theorem on the example of rooted trees.

Example (rooted trees [20, Sec. 3.1])**.** Let $\hat{T}(x)$ denote the counting series for rooted trees. We can form a rooted tree which has degree n on its root vertex by taking a (root) vertex and concatenating it with the root-vertices of n rooted trees (see Figure 1.1). How many different trees can we obtain by this method?

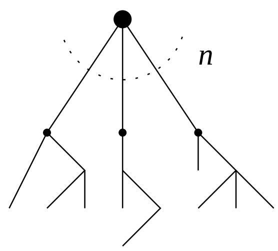

Figure 1.1: Construction of a rooted tree by joining n rooted tree.

Consider the group $A = S_n$ with object set $X = \{1, \ldots, n\}$ and the group $B = E$ (identity group) having the rooted trees as object set Y. Then each function $f \in Y^X$ corresponds to a uniquely determined rooted tree with degree n at its root vertex (as constructed above). We define $w(y)$ to be the number of vertices in the tree $y \in Y$.

The series $\hat{T}(x) = \sum_n \hat{t}_n x^n$ enumerates the elements of Y by weight, i.e., it is the "figure counting series". The weight of $f \in Y^X$, $w(f) = \sum_{x \in X} w(f(x)) = \sum_{i=1}^n w(f(i))$, is the total number of vertices minus one in the constructed rooted tree, since it is not counting the new root vertex.

For the symmetric group S_n, the orbits of E^{S_n} corresponds to rooted trees with degree n at the root vertex. Thus by Pólya's theorem (Thm. 1.9) $Z\left(S_n; \hat{T}(x)\right)$ is the function counting series, but it is not counting the root node. Hence, $xZ\left(S_n; \hat{T}(x)\right)$ counts trees with degree n on its root node and so, using (1.1),

$$\hat{T}(x) = x \sum_{n=0}^{\infty} Z\left(S_n; \hat{T}(x)\right) = x \exp\left\{\sum_{k=1}^{\infty} \frac{1}{k} \hat{T}\left(x^k\right)\right\}. \tag{1.5}$$

1.2.3 Composition theorem

Using Pólya's theorem, we obtain the generating function for a composition of objects with known generating functions where similarities between composed objects were induced by the composition only. Next, we generalize this method to compute the sum of cycle indices of configurations given the cycle index of the composition and the cycle indices of the figures. By use of Pólya's theorem, the computed cycle index sum can then be easily converted to a generating function of the configurations.

Let a permutation group A on $X = \{1, \ldots, n\}$, a set $Y \subseteq \mathbb{N}$, and object sets Y_i with permutation groups B_i for each $i \in Y$ be given. For an example, each $i \in Y$ might correspond to a graph on the point set Y_i. Assume further that only finitely many Y_i have the same cardinality so that the sum $\sum_{i \in Y} Z(B_i)$ is defined.

A composition of n objects from Y is given by a function $f : X \to Y$. We let $A(f) := \{\alpha \in A \,|\, f(x) = f(\alpha x), \forall x \in X\}$ be the subgroup of A of those permutations $\alpha \in A$ for which f is constant on the cycles of α. When we let $X_i := f^{-1}(i)$ for each $i \in Y$, we have $\alpha(X_i) = X_i$ for all $i \in Y$ and $\alpha \in A(f)$.

If $f(X) = \{i_1, \ldots, i_m\}$ we let $A(f)[B_{i_1}, \ldots, B_{i_m}]$ be the *composition group* with object set $\bigcup_{i \in Y}(X_i \times Y_i)$, defined by

$$A(f)[B_{i_1}, \ldots, B_{i_m}] := \{(\alpha; \tau_1, \ldots, \tau_n) \,|\alpha \in A(f), \tau_j : X_j \to B_j, j = 1, \ldots, n\}$$

and

$$(\alpha; \tau_1, \ldots, \tau_n)(x, y) := (\alpha x, \tau_i(x) y), \qquad (x, y) \in (X_i, Y_i), \ i \in Y.$$

Intuitively, $A(f)[B_{i_1}, \ldots, B_{i_m}]$ permutes copies of the sets Y_{i_k} for $k = 1, \ldots, m$, where each element of X_{i_k} receives one copy of Y_{i_k}. The copies of Y_{i_k} are permuted within themselves by the permutations in $A(f)$, while the elements of each copy are permuted independently by B_{i_k}.

We say that two functions f and g are *similar*, $f \sim g$, iff there exists a permutation $\alpha \in A$ such that $f(x) = g(\alpha x)$ for all $x \in X$, that is, f and g are in the same orbit of E^A (See Figure 1.2 as an example.). Denote the set of orbits of E^A by $\mathcal{F}$. If $f \sim g$, i.e., f and g share the same orbit, then $A(f)$ and $A(g)$ are conjugate and furthermore, $A(f)[B_{i_1}, \ldots, B_{i_m}]$ and $A(g)[B_{i_1}, \ldots, B_{i_m}]$ are also conjugate. Hence, they have the same cycle index, and we can define the cycle index of any equivalence class of functions $F \in \mathcal{F}$ to be the cycle index of the composition group determined by some function $f \in F$, $Z(F) := Z(f)$ for any $f \in F$. Now let $Z(\mathcal{F}) := \sum_{F \in \mathcal{F}} Z(F)$. Then the following theorem relates $Z(\mathcal{F})$ to $Z(A)$ and the $Z(B_i)$, $i \in Y$.

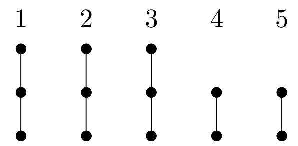

Figure 1.2: A composition of paths on 3 and 2 vertices. We let $Y = \{1, 2\}$ with Y_1 the path on 3 vertices and Y_2 the path on 2 vertices. Let $X = \{1, 2, 3, 4, 5\}$ and $A = \{\mathrm{id}, (1\,2)(3\,4)(5)\}$. The composition in this figure corresponds to the function $f = (1, 1, 1, 2, 2)$ (i.e., $f(1) = f(2) = f(3) = 1$, $f(4) = f(5) = 2$), and it is $A(f) = \{\mathrm{id}\}$, $X_1 = \{1, 2, 3\}$ and $X_2 = \{4, 5\}$. The function $g = (1, 1, 2, 1, 2)$ is similar to f, $f \sim g$, since $f = g\alpha$ for $\alpha = (1\,2)(3\,4)(5)$.

Theorem 1.11 (composition [20, page 182])**.**

$$Z(\mathcal{F}) = Z(A)\left[\sum_{i \in Y} Z(B_i)\right],$$

where $Z(A)\left[\sum_{i \in Y} Z(B_i)\right]$ denotes the formal power series obtained from $Z(A)$ when each variable s_k is replaced by $\sum_{i \in Y} Z(B_i; s_k, s_{2k}, s_{3k}, \ldots)$.

The sum $Z(\mathcal{F}) = \sum_{F \in \mathcal{F}} Z(F)$ is called *cycle index sum*. It counts the permutations for all configurations that can be obtained by composing the objects Y_i with groups B_i along the objects X with group A. Application of Pólya's theorem with figure counting series $c(x) := x$ to a configuration $F \in \mathcal{F}$ gives $Z(F; x) = x^p$ with p the order of the composition F, that is, $p =$

$\sum_{k=1}^{m} |Y_{i_k}|$ for $f(X) = \{i_1, \ldots, i_m\}$, $f \in F$. Hence, summing over all configurations gives the configuration counting series $\mathcal{F}(x)$ for $\mathcal{F}$:

$$\mathcal{F}(x) = Z(\mathcal{F}; x) := \sum_{F \in \mathcal{F}} Z(F; x).$$

1.2.4 Rooted connected graphs with given blocks

For a graph G on n labelled vertices (i.e., object set $X = \{1, \ldots, n\}$) we define the permutation group of G to consist of all permutations on X which preserve adjacency in G, i.e., for each edge $\{i, j\}$ in G and each permutation α in the permutation group of G, $\{\alpha(i), \alpha(j)\}$ is also an edge in G. Denote the permutation group of G by $\Gamma(G)$. We say two graphs G_1 and G_2 with the same object set X are *isomorphic* if there is a permutation $\alpha \in \Gamma(G_1)$ which maps G_1 onto G_2, i.e., $\{i, j\} \in G_1 \Leftrightarrow \{\alpha(i), \alpha(j)\} \in G_2$ for all $i, j \in X$. This relation is again an equivalence relation, its orbits are the unlabelled graphs.

For a set $\mathcal{G}$ of graphs, denote by $Z(\mathcal{G}) := \sum_{G \in \mathcal{G}} Z(G)$ the *cycle index sum* of $\mathcal{G}$, with $Z(G) := Z(\Gamma(G))$ the cycle index of (the group) of the graph.

Definition 1.12 (rooted graph)**.** A *rooted graph* (G, u) is a graph G together with a distinguished vertex u. Two rooted graphs (G_1, u_1) and (G_2, u_2) are called isomorphic when there is a permutation $\alpha \in \Gamma(G_1)$ that maps G_1 onto G_2 and u_1 onto u_2. The group of (G, u) consists of all permutations in $\Gamma(G)$ which fix u.

Similarly, we say a graph is rooted at an edge (or *edge rooted*) when we distinguish one of its edges from the others. The same applies to *face rooted* graphs. Note that we do orient the root-edge or root-face.

For a set of graphs $\mathcal{G}$ we denote by $\hat{\mathcal{G}}$ the graphs which can be obtained from $\mathcal{G}$ by selecting a vertex as a root.

Definition 1.13 (block)**.** A *cutvertex* of a graph G is a vertex whose removal increases the number of components of G. A *block* is an inclusion-maximal connected subgraph of G without a cutvertex. (This includes edges as well.)

We proceed now with a derivation of an implicit formula for the cycle index sum for rooted connected graphs with given blocks.

Let $\mathcal{D}$ be a set of blocks and $\hat{\mathcal{D}}$ the corresponding rooted blocks. Let $\hat{\mathcal{C}}$ be the set of rooted connected graphs whose blocks are in $\hat{\mathcal{D}}$. The graphs in $\hat{\mathcal{C}}$ which are rooted at a vertex which is not a cutvertex can be constructed by taking a rooted block D from $\hat{\mathcal{D}}$ and attaching rooted connected graphs from $\hat{\mathcal{C}}$ at each vertex of the block, except for the root vertex (see Figure 1.3, left picture). Let X be the non-root vertices of D and $Z(A) = \frac{1}{s_1} Z(\Gamma(D))$ be the permutation group of A, where the factor $\frac{1}{s_1}$ is due to the exclusion of the root vertex. Using the composition theorem 1.11 we obtain

$$s_1 \left(\frac{Z\left(\hat{\mathcal{D}}\right)}{s_1} \right) \left[Z\left(\hat{\mathcal{C}}\right) \right] \tag{1.6}$$

as cycle index sum for connected graphs rooted at a non-cutvertex and blocks in $\hat{\mathcal{D}}$. Next, the cycle index sum for rooted connected graphs where the root vertex is incident with exactly n blocks (see Figure 1.3, right picture), $n \geq 2$, can be obtained by another application of the composition theorem. We compose n non-cutvertex rooted connected graphs by joining them at their root vertex, i.e., we let $X = \{1, \ldots, n\}$, $A = S_n$, and Y be the set of non-cutvertex rooted connected graphs excluding the root vertex from the counting, which are counted by Formula (1.6), divided by s_1. Hence,

$$s_1 Z(S_n)\left[\left(\frac{Z(\hat{\mathcal{D}})}{s_1}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right]\right]$$

is the cycle index sum for rooted connected graphs where the root vertex is incident with exactly n blocks, and the additional factor s_1 is due to the root vertex of the composed graph.

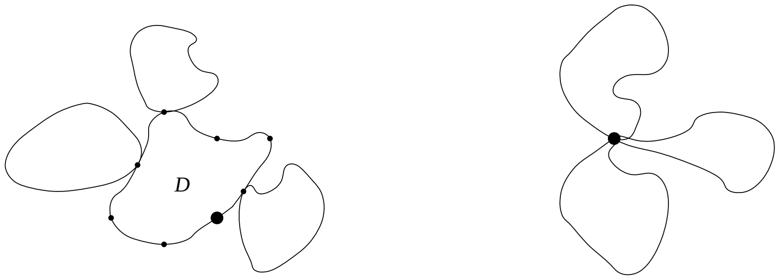

Figure 1.3: Construction of a rooted connected graph from its blocks. Left: Rooted at a vertex which is not a cutvertex. Right: Rooted at a cutvertex.

Summing over all $n \geq 1$, and inclusion of the term for the single vertex, yields

$$\begin{aligned} Z\left(\hat{\mathcal{C}}\right) &= s_1 + s_1\left(\frac{Z(\hat{\mathcal{D}})}{s_1}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right] + \sum_{n\geq 2} s_1 Z(S_n)\left[\left(\frac{Z(\hat{\mathcal{D}})}{s_1}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right]\right] \\ &= s_1 \sum_{n\geq 0} Z(S_n)\left[\left(\frac{Z(\hat{\mathcal{D}})}{s_1}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right]\right], \end{aligned}$$

and by using (1.1) this gives

$$Z\left(\hat{\mathcal{C}}\right) = s_1 \exp\left(\sum_{k\geq 1}\frac{s_k}{k}\left[\left(\frac{Z(\hat{\mathcal{D}})}{s_1}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right]\right]\right). \tag{1.7}$$

Due to Möbius inversion[1] (1.7) becomes

$$\left(\frac{Z(\hat{\mathcal{D}})}{s_1}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right] = \sum_{k\geq 1}\frac{\mu(k)}{k}\log\left(s_k\left[\frac{Z\left(\hat{\mathcal{C}}\right)}{s_1}\right]\right). \tag{1.8}$$

1.2.5 Unrooting graphs

When enumerating labelled graphs, it is known that the exponential generating function for the rooted graphs can be obtained from the exponential generating function of the unrooted graphs by differentiation with respect to the formal variable [20, page 10]. It turns out that a similar relation holds for the cycle index sums of rooted and unrooted unlabelled graphs. Observe that

[1]The Möbius Function $\mu : \mathbb{N} \to \mathbb{Z}$ satisfies $\mu(n) = (-1)^r$ if n is a product of r distinct primes and $\mu(n) = 0$ if n is divisible by a square. The Möbius Inversion Formula [35] inverts the sequence $a(n) \equiv \sum_{d|n} b(d)$ to $b(n) \equiv \sum_{d|n}\mu(d)\, a\left(\frac{n}{d}\right)$.

the group of a rooted graph consists of those permutations from the group of the unrooted graphs which fix the root vertex. Permutations which permute all vertices of an unrooted graph, i.e., have no cycle of length one, can therefore not belong to one of the permutation groups of the rooted graphs. Furthermore, the remaining permutations, which keep at least one vertex fixed, can be divided into the permutations of the rooted graphs. More formally, we can prove the following theorem.

Theorem 1.14 ([20, page 185]). *Let $\mathcal{G}$ be a set of graphs and $\hat{\mathcal{G}}$ its rooted companion. Then*

$$Z\left(\hat{\mathcal{G}}\right) = s_1 \frac{\partial Z(\mathcal{G})}{\partial s_1}.$$

Proof. For a graph $G \in \mathcal{G}$, let $A := \Gamma(G)$ be the group of G and X the vertices of G. For $\alpha \in A$ let $\psi(\alpha) := \prod_l s_l^{j_l(\alpha)}$, and thus by Definition 1.1 $Z(A) = \frac{1}{|A|}\sum_{\alpha \in A}\psi(\alpha)$. For vertices $y_1, \ldots, y_m$, $m := N(A)$, from each orbit of G let $A_k := \{\alpha \in A \,|\, \alpha y_k = y_k\}$ be the permutations on G which fix y_k (stabilizer of y_k). That is, A_k is the group of $G_k \in \hat{\mathcal{G}}$, where G_k is obtained from G by rooting at the orbit that contains y_k, $k = 1, \ldots, m$. From Lemma 1.4 we know $|A| = |A_k|\,|X_k|$.

Let $X_1, \ldots, X_m$ denote the orbits of X and for $x \in X$ let $k(x) := k$ such that $x \in X_k$ and $\sigma_x \in A$ such that $\sigma_x x = y_{k(x)}$. Observe that $\psi(\alpha) = \psi\left(\sigma^{-1}\alpha\sigma\right)$ for all $\alpha, \sigma \in A$. Now,

$$\begin{aligned}
|A| \sum_{k=1}^{m} Z(A_k) &= |A| \sum_{k=1}^{m} \frac{1}{|A_k|} \sum_{\alpha \in A_k} \psi(\alpha) \\
&= \sum_{k=1}^{m} |X_k| \sum_{\alpha \in A_k} \psi(\alpha) \\
&= \sum_{x \in X} \sum_{\alpha \in A_{k(x)}} \psi(\alpha) \\
&= \sum_{\alpha \in A} \sum_{\substack{x \in X: \\ \alpha(\sigma_x x) = \sigma_x x}} \psi(\alpha) \\
&= \sum_{\alpha \in A} \sum_{\substack{x \in X: \\ \alpha x = x}} \psi\left(\sigma_x^{-1}\alpha\sigma_x\right) \\
&= \sum_{\alpha \in A} j_1(\alpha)\,\psi(\alpha) \\
&= |A|\, s_1 \frac{\partial}{\partial s_1} Z(A).
\end{aligned}$$

□

Applying this theorem inversely we can obtain the cycle index sum of the unrooted graphs from the rooted graphs by integration. Recall that when a graph is rooted, all permutations that fix no vertices are "lost". This corresponds to the additional boundary term in the following formula for $Z(\mathcal{G})$:

$$Z(\mathcal{G}) = \int_0^{s_1} \frac{Z\left(\hat{\mathcal{G}}\right)}{s_1} ds_1 + Z(\mathcal{G})\,|_{s_1=0}\,. \tag{1.9}$$

1.2.5.1 Unrooting vertex-rooted graphs with given blocks

When we apply formula (1.9) to the connected graphs $\hat{\mathcal{C}}$ to obtain the cycle index sum of the corresponding unlabelled connected graphs $\mathcal{C}$ we obtain

$$Z(\mathcal{C}) = \int_0^{s_1} \frac{Z\left(\hat{\mathcal{C}}\right)}{s_1} ds_1 + Z(\mathcal{C})\,|_{s_1=0}\,.$$

For the boundary term, which counts fixed-vertex free permutations, the following observation is helpful.

Theorem 1.15 ([20, page 186]). *Every connected graph G which has a fixed-vertex free permutation has a unique block whose vertices are permuted among themselves by all permutations in the group of G.*

Therefore, the graphs counted by the term $Z(\mathcal{C})|_{s_1=0}$ can be constructed by rooting at a block. That is, we take an unrooted block as root block and attach vertex-rooted connected graphs at all its vertices. After this, we have to remove those permutations which are not fixed-vertex free:

$$Z(\mathcal{C})|_{s_1=0} = Z(\mathcal{D})\left[Z\left(\hat{\mathcal{C}}\right)\right]|_{s_1=0} = Z(\mathcal{D})|_{s_1=0}\left[Z\left(\hat{\mathcal{C}}\right)\right].$$

This expresses the cycle index sum of $\mathcal{C}$ by the cycle index sums of $\mathcal{D}$ and $\hat{\mathcal{C}}$:

$$Z(\mathcal{C}) = \int_0^{s_1} \frac{Z\left(\hat{\mathcal{C}}\right)}{s_1} ds_1 + Z(\mathcal{D})|_{s_1=0}\left[Z\left(\hat{\mathcal{C}}\right)\right]. \tag{1.10}$$

In general, the integration in (1.9) might not be easy to handle. In our case, $Z\left(\hat{\mathcal{C}}\right)$ is only implicitly defined by (1.7) or (1.8), but it can be used to solve the integration in (1.10), as shown next. We remark here a typographic error in our edition of [20]. Formula (8.7.10) should have a $[s_1 Z(\mathcal{K}')]$ at its end.

Theorem 1.16 ([20, page 190]). *The cycle index sum $Z(\mathcal{C})$ for connected graphs whose blocks are in the set $\mathcal{D}$ is given by*

$$Z(\mathcal{C}) = Z\left(\hat{\mathcal{C}}\right) - Z\left(\hat{\mathcal{D}}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right] + Z(\mathcal{D})\left[Z\left(\hat{\mathcal{C}}\right)\right] \tag{1.11}$$

Proof. We prove equation (1.11) by showing that formula (1.10) is satisfied. Observe from (1.8) that

$$\begin{aligned}
\frac{\partial}{\partial s_1}\left(Z\left(\hat{\mathcal{D}}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right]\right) &= \frac{\partial}{\partial s_1}\left(Z\left(\hat{\mathcal{C}}\right)\sum_{k\geq 1}\frac{\mu(k)}{k}\log\left(s_k\left[\frac{Z\left(\hat{\mathcal{C}}\right)}{s_1}\right]\right)\right)\\
&= \left\{\frac{\partial}{\partial s_1}Z\left(\hat{\mathcal{C}}\right)\right\}\left(\frac{Z\left(\hat{\mathcal{D}}\right)}{s_1}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right] + Z\left(\hat{\mathcal{C}}\right)\frac{\partial}{\partial s_1}\log\left(\frac{Z\left(\hat{\mathcal{C}}\right)}{s_1}\right)\\
&= \left\{\frac{\partial}{\partial s_1}Z\left(\hat{\mathcal{C}}\right)\right\}\left(\frac{Z\left(\hat{\mathcal{D}}\right)}{s_1}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right] + Z\left(\hat{\mathcal{C}}\right)\frac{s_1}{Z\left(\hat{\mathcal{C}}\right)}\frac{\partial}{\partial s_1}\left(\frac{Z\left(\hat{\mathcal{C}}\right)}{s_1}\right)\\
&= \left\{\frac{\partial}{\partial s_1}Z\left(\hat{\mathcal{C}}\right)\right\}\left(\frac{Z\left(\hat{\mathcal{D}}\right)}{s_1}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right] + s_1\left(\frac{s_1\frac{\partial}{\partial s_1}Z\left(\hat{\mathcal{C}}\right) - Z\left(\hat{\mathcal{C}}\right)}{s_1^2}\right)\\
&= \left\{\frac{\partial}{\partial s_1}Z\left(\hat{\mathcal{C}}\right)\right\}\left(\frac{Z\left(\hat{\mathcal{D}}\right)}{s_1}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right] + \frac{\partial}{\partial s_1}Z\left(\hat{\mathcal{C}}\right) - \frac{Z\left(\hat{\mathcal{C}}\right)}{s_1}.
\end{aligned}$$

In addition, we observe that the following "chain rule" holds:

$$\frac{\partial}{\partial s_1}Z(\mathcal{D})\left[Z\left(\hat{\mathcal{C}}\right)\right] = \left(\frac{\partial}{\partial s_1}Z(\mathcal{D})\right)\left[Z\left(\hat{\mathcal{C}}\right)\right]\frac{\partial}{\partial s_1}Z\left(\hat{\mathcal{C}}\right).$$

Putting these facts together we have

$$\frac{\partial}{\partial s_1}\left(Z\left(\hat{\mathcal{C}}\right) - Z\left(\hat{\mathcal{D}}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right] + Z(\mathcal{D})\left[Z\left(\hat{\mathcal{C}}\right)\right]\right)$$
$$= -\frac{\partial}{\partial s_1}Z\left(\hat{\mathcal{C}}\right)\left(\frac{Z\left(\hat{\mathcal{D}}\right)}{s_1}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right] + \frac{Z\left(\hat{\mathcal{C}}\right)}{s_1} + \left(\frac{\partial}{\partial s_1}Z(\mathcal{D})\right)\left[Z\left(\hat{\mathcal{C}}\right)\right]\frac{\partial}{\partial s_1}Z\left(\hat{\mathcal{C}}\right) = \frac{Z\left(\hat{\mathcal{C}}\right)}{s_1},$$

where the last equality is due to $Z\left(\hat{\mathcal{D}}\right) = s_1\frac{\partial}{\partial s_1}Z(\mathcal{D})$ from Theorem 1.14.

It remains to check the boundary condition:

$$\left(Z\left(\hat{\mathcal{C}}\right) - Z\left(\hat{\mathcal{D}}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right] + Z(\mathcal{D})\left[Z\left(\hat{\mathcal{C}}\right)\right]\right)\Big|_{s_1=0} = Z(\mathcal{D})\left[Z\left(\hat{\mathcal{C}}\right)\right]\Big|_{s_1=0}. \qquad \square$$

In the light of the formulas (1.7) and (1.11) we see how the computation of the cycle index sum for connected graphs with given blocks can be reduced to the derivation of the cycle index sums of the rooted and unrooted blocks.

Remark. A more intuitive access to Theorem 1.16 might be the following: The permutation group of a connected graph can be divided into those permutations which fix a vertex and those permutations which have no fixed vertex. The former can be identified with vertex rooted graphs, thereby the term $Z\left(\hat{\mathcal{C}}\right)$. By Theorem 1.15, the latter permutations fix a block. Hence, they can be identified by block rooted graphs which have no fixed vertex. Finally, graphs which fix a block but not a vertex can be counted by consideration of block rooted graphs ($Z(\mathcal{D})\left[Z\left(\hat{\mathcal{C}}\right)\right]$) and substraction of graphs which are rooted at an block which is itself rooted at a vertex ($Z\left(\hat{\mathcal{D}}\right)\left[Z\left(\hat{\mathcal{C}}\right)\right]$).

Example (unrooted trees)**.** Let $\mathcal{D}$ consist of the graph which has a single edge only. Then $Z(\mathcal{D}) = \frac{1}{2}\left(s_1^2 + s_2\right)$, $Z\left(\hat{\mathcal{D}}\right) = s_1^2$, $\mathcal{C}$ is the set of trees $\mathcal{T}$, and we obtain from (1.11),

$$Z(\mathcal{T}) = Z\left(\hat{\mathcal{T}}\right) + \frac{1}{2}\left(s_1^2 + s_2\right)\left[Z\left(\hat{\mathcal{T}}\right)\right] - s_1^2\left[Z\left(\hat{\mathcal{T}}\right)\right] = Z\left(\hat{\mathcal{T}}\right) + \frac{1}{2}\left(Z\left(\hat{\mathcal{T}}; s_2\right) - Z\left(\hat{\mathcal{T}}\right)^2\right),$$

which yields

$$T(x) = \hat{T}(x) + \frac{1}{2}\left(\hat{T}\left(x^2\right) - \hat{T}(x)^2\right) \tag{1.12}$$

as the ordinary generating function of unlabelled unrooted trees, which one can also find in [20, Sec. 3.2].

1.2.6 Unrooting trees

In addition to the above formula for the cycle index sum of unrooted trees, we can also generalize the well-known dissimilar characteristic theorem for trees. This will turn out to be useful for the cycle index sum of unrooted dissections in the next chapter.

We call an edge in a tree a *symmetry edge* if the group of G contains a permutation that permutes the vertices of this edge. Two vertices are called *dissimilar* when they are not similar, i.e., there is no permutation in $\Gamma(G)$ that maps one vertex onto the other. The number of dissimilar vertices is the number of orbits in $\Gamma(G)$. Similarly, two edges are called dissimilar when there is no permutation in $\Gamma(G)$ that maps one edge onto the other.

Theorem 1.17 (dissimilar characteristic theorem for trees [20, page 56])**.** *Let G be a tree. Denote by n^* the number of dissimilar vertices in G, by m^* the number of dissimilar edges and by s the number of symmetry edges in G. Then*

$$1 = n^* - m^* + s. \tag{1.13}$$

Proof. We prove this formula by induction over m^*. The case $m^* = 0$ is trivial. If $m^* = 1$ the tree has to be either a star graph on at least 3 vertices or consists of a single edge only. The former has no symmetry edges but two orbits of vertices, the latter has a symmetry edge and only one orbit of vertices.

For $m^* \geq 2$ pick a leaf-edge. After removing all edges which are similar to this edge and all vertices which are similar to the leaf, the number of dissimilar vertices and edges reduces by 1, respectively. The number of symmetry edges remains unchanged. Application of the induction hypothesis finishes the proof. □

Formula (1.12) can also be derived from this theorem by summing the left and right side of (1.13) over unlabelled trees and correct identification of the resulting counting serieses. But when we are interested in the cycle index sum of unlabelled trees, we have to be more careful. Observe that in the proof of the above theorem the symmetry edge plays only a role in the induction start when we consider the tree consisting of a single edge only. The cycle index of this single edge is $\frac{1}{2}\left(s_1^2 + s_2\right)$, when rooting at a vertex it is s_1^2 and when rooting at a (symmetry) edge it is $\frac{1}{2}\left(s_1^2 + s_2\right)$, so just subtracting the cycle index of the edge-rooted graph from the vertex-rooted ones and adding the cycle index of the symmetry edge rooted graph does not give the correct result here. But with a simple variation, we can fix this problem. For a tree G we let $\mathcal{V}$ be the set of trees which can be obtained from G by rooting at a vertex. We divide this set into the set of rooted trees $\mathcal{V}_1$ where the root vertex is not incident to a symmetry edge and the set $\mathcal{V}_2$ where the root vertex is incident to a symmetry edge. Furthermore, we let $\mathcal{E}$ be the trees obtained from G by rooting at an edge, and let $\mathcal{S}$ be the trees obtained from G by rooting at a symmetry edge.

Theorem 1.18 (dissimilar characteristic theorem for trees, cycle index version)**.**

$$Z(G) = Z(\mathcal{V}_1) - Z(\mathcal{E}) + 2Z(\mathcal{S}). \tag{1.14}$$

Remark. Trees which are rooted at a vertex that is incident to a symmetry edge ($\mathcal{V}_2$) are in a one to one correspondence to the trees in $\mathcal{S}$, the only difference is its cycle index. After application of the substitution $s_i \mapsto x^i$ this difference disappears and we obtain again formula (1.13).

Proof. We prove the theorem by induction over the number of dissimilar edges: If there is only one dissimilar edge, the tree is a star-graph $K_{1,k}$, $k \geq 1$. For $k = 1$, G consists of one edge, which is a symmetry edge. So $Z(G) = \frac{1}{2}\left(s_1^2 + s_2\right)$, $Z(\mathcal{V}_1) = 0$, $Z(\mathcal{E}) = Z(\mathcal{S}) = \frac{1}{2}\left(s_1^2 + s_2\right)$.

For $k \geq 2$, G has no symmetry edge. The permutations in $\Gamma(G)$ have to fix the vertex with degree k and can permute the remaining k vertices arbitrarily. Thus $Z(G) = s_1 Z(S_k)$. For the vertex rooted graphs, there are now two possibilities. Rooting at the center gives $s_1 Z(S_k)$, rooting at a leaf gives $s_1^2 Z(S_{k-1})$, so $Z(\mathcal{V}_1) = s_1 Z(S_k) + s_1^2 Z(S_{k-1})$. For edge-rooted trees we obtain $Z(\mathcal{E}) = s_1^2 Z(S_{k-1})$ and $\mathcal{S}$ is empty in this case.

Now let G be a tree with at least two dissimilar edges. Denote by X the set of vertices of G and let $A := \Gamma(G)$. Let Y be an orbit of leaves. Observe that the neighborhood $\delta(Y)$ is also an orbit. Let $m := |\delta(Y)|$. Consider the subgraph H of G induced by the neighborhood of a vertex $x \in \delta(Y)$ with respect to Y, $H := G[\delta(x) \cap Y]$. Since Y forms an orbit, H is independent of the choice of $x \in \delta(Y)$. Let G' be the tree obtained from G by removing the vertices Y. Let $A' := \Gamma(G')$. Then we identify with each permutation $\alpha \in A$ the pair (α', τ) with $\alpha' := \alpha\big|_{X \setminus Y} \in A'$ and $\tau : \delta(Y) \to \Gamma(H)$, $\tau(x) := \alpha\big|_{\delta(x) \cap Y}$ for $x \in \delta(Y)$. On the other hand, each pair $(\alpha', \tau) \in A' \times \Gamma(H)^{\delta(Y)}$ can be identified with a permutation $\alpha \in A$ by $\alpha(x) := \alpha'(x)$ for $x \in X \setminus Y$ and $\alpha(y) := \tau(\delta(y))\, y$ for $y \in Y$. Therefore, $|A| = |A'|\, |\Gamma(H)|^m$.

Letting $\psi(\alpha) := \prod_{k=1}^{|X|} s_k^{j_k(\alpha)}$ again, we have

$$\begin{aligned}
Z(A) &= \frac{1}{|A|}\sum_{\alpha\in A}\psi(\alpha)\\
&= \frac{1}{|A'|\,|\Gamma(H)|^m}\sum_{\alpha'\in A',\tau:\delta(Y)\to\Gamma(H)}\psi(\alpha')\prod_{x\in\delta(Y)}\psi(\tau(x))\\
&= \frac{1}{|A'|}\left(\sum_{\alpha'\in A'}\psi(\alpha')\right)\frac{1}{|\Gamma(H)|^m}\sum_{\beta_1,\ldots,\beta_m\in\Gamma(H)}\prod_{i=1}^{m}\psi(\beta_i)\\
&= \frac{1}{|A'|}\left(\sum_{\alpha'\in A'}\psi(\alpha')\right)\left(\frac{1}{|\Gamma(H)|}\sum_{\beta\in\Gamma(H)}\psi(\beta)\right)^m\\
&= Z(A')\,Z(H)^m.
\end{aligned}$$

For $x \in X$ not incident to a symmetry edge let G_x be the graph obtained from G by rooting at vertex x. If $x \notin Y$ let G'_x be the graph obtained from G_x by removing the vertices in Y. Then, similarly as above, $Z(G_x) = Z(G'_x)\,Z(H)^m$. For an edge e denote by G_e the graph obtained from G by rooting at the edge e and if e is not incident to Y let G'_e be the graph obtained from G_e by removing the vertices in Y. Then $Z(G_e) = Z(G'_e)\,Z(H)^m$. For an edge e which is incident to Y the cycle index coincides with the one of $Z(G_{e\cap Y})$, since rooting at a leaf-edge fixes the leaf and rooting at a leaf fixes the leaf edge. Finally, observe that such a leaf edge cannot be a symmetry edge.

For G' let now $\mathcal{V}'_1$, $\mathcal{E}'$, and $\mathcal{S}'$ be the graphs obtained from G' by rooting at a vertex that is not incident to a symmetry edge, an edge, and a symmetry edge, respectively. Then from the induction hypothesis

$$Z(\mathcal{G}') = Z(\mathcal{V}'_1) - Z(\mathcal{E}') + 2Z(\mathcal{S}').$$

Now, for $y \in Y$,

$$Z(\mathcal{V}'_1) = \sum_{x\notin Y} Z(G'_x) = \sum_{x\notin Y} Z(G_x)\,Z(H)^{-m} = (Z(\mathcal{V}_1) - Z(G_y))\,Z(H)^{-m},$$

where the sums are over all dissimilar vertices $x \notin Y$. Similarly, for e incident to Y,

$$Z(\mathcal{E}') = \sum_{f\not\sim e} Z\left(G'_f\right) = \sum_{f\not\sim e} Z(G_f)\,Z(H)^{-m} = (Z(\mathcal{E}) - Z(G_e))\,Z(H)^{-m},$$

where the sums are over all dissimilar edges f which are not similar to e. Finally, $Z(\mathcal{S}') = Z(\mathcal{S})\,Z(H)^{-m}$. So, since $Z(G_e) = Z(G_y)$ and $Z(G) = Z(G')\,Z(H)^m$, we can prove

$$Z(G)\,Z(H)^{-m} = Z(\mathcal{V}_1)\,Z(H)^{-m} - Z(\mathcal{E})\,Z(H)^{-m} + 2Z(\mathcal{S})\,Z(H)^{-m}. \qquad \square$$

1.3 Singularity analysis and asymptotic enumeration

After determination of an explicit or implicit equation for a counting series, one is interested in properties of its coefficients. Closed formulas for their exact numbers are often not easy to obtain. Instead we compute asymptotic estimates.

Until now, cycle index sums and generating functions have been treated as formal power serieses. But when we look at these functions from the point of complex analysis, we can retrieve a lot of information about its coefficients, or at least of an asymptotic estimate. Next, we will show how the theory of analytic functions and the analysis of singularities can be used to determine such information. The material for this section is based on [16].

Definition 1.19 (analytic, radius of convergence)**.** A function $f : \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}$, is *analytic at a point* $z_0 \in \Omega$ if for z in some open disc centered at z_0 and contained in Ω, f is representable by a convergent power series,

$$f(z) = \sum_{n \geq 0} c_n (z - z_0)^n .$$

The *radius of convergence* is the radius of the greatest disc centered at z_0 and contained in Ω that permits a convergent power series expansion.

A function is *analytic in a region* Ω iff it is analytic at every point in Ω.

Notation. For a function $f(z)$ which is analytic at 0 we write $[z^n] f(z)$ for the coefficient of z^n in the power series expansion of $f(z)$ around 0.

Definition 1.20 (complex-differentiable)**.** A function $f : \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}$, is called *complex-differentiable* (or *holomorphic*) at z_0 if, for complex w, the limit

$$\lim_{w \to 0} \frac{1}{w} (f(z_0 + w) - f(z_0))$$

exists. This limit is denoted by $f'(z_0)$ or $\frac{d}{dz} f(z)|_{z=z_0}$. A function is *complex-differentiable* in Ω iff it is differentiable at every $z_0 \in \Omega$.

It is well known that analyticity and complex differentiability are equivalent:

Theorem 1.21 ([16, page 218])**.** *A function is analytic in a region Ω if and only if it is complex-differentiable in Ω.*

The concept of analyticity can be extended to meromorphic functions by allowing also negative exponents in the power series expansion:

Definition 1.22 (meromorphic, pole, residue)**.** A function $h(z)$ is *meromorphic at a point* z_0 iff in a neighborhood of z_0 it is representable by an expansion of the form

$$h(z) = \sum_{n \geq -M} h_n (z - z_0)^n$$

for $z \neq z_0$. If $h_{-M} \neq 0$, then $h(z)$ is said to have a *pole* of order M at z_0. The coefficient h_{-1} is called the *residue* of $h(z)$ at z_0 . A function is *meromorphic in a region* iff it is meromorphic at any point of the region.

The next theorems relate global properties of holomorphic and meromorphic functions to the coefficients in their power series expansion.

Theorem 1.23 ([16, Thm. IV.2])**.** *Let f be analytic in Ω and let λ be a closed simple curve in Ω. Then*

$$\int_\lambda f = 0.$$

Theorem 1.24 (Cauchy's residue theorem [16, Thm IV.3])**.** *Let $h(z)$ be meromorphic at a point z_0 and let λ be a simple loop in the neighborhood of z_0 in which h permits a power series expansion as in Definition 1.22. Then*

$$\frac{1}{2\pi i} \int_\lambda h(z)\, dz = h_{-1}.$$

Corollary 1.25 (Cauchy's coefficient formula [16, Thm. IV.4])**.** *Let $f(z)$ be analytic at a point z_0 and let λ be a simple loop in the neighborhood of z_0 in which $f(z)$ permits a power series expansion as in Definition 1.19. Then*

$$f_n = \frac{1}{2\pi i} \int_\lambda \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

We proceed now with the definition of singularities and show how their location is related to the coefficients of the power series.

Definition 1.26 (analytically continuable, singularity)**.** Let $f(z)$ be an analytic function defined over the interior region determined by a simple closed curve γ, and let z_0 be a point on γ. If there exists an analytic function $f^*(z)$ defined over some open set Ω^* containing z_0 and coinciding with $f(z)$ in $\Omega \cap \Omega^*$, then $f(z)$ is called *analytically continuable* at z_0. If f is not analytically continuable at z_0, then z_0 is called a *singular point* or *singularity* of f.

Theorem 1.27 (boundary singularities [16, Thm. IV.5, IV.6])**.** *A function $f(z)$ which is analytic at z_0 with radius of convergence $R < \infty$ has a singularity on the boundary of its disc of convergence.*

If in addition $z_0 = 0$ and f has nonnegative Taylor coefficients, then the point $z = R$ is a singularity of f.

For a function $f(z)$ which is analytic around 0 the singularities on the boundary of its disc of convergence are called *dominant singularities*. The next theorem shows that they determine the asymptotics of the coefficients of $f(z)$.

Theorem 1.28 (exponential growth formula [16, Thm. IV.7])**.** *If $f(z)$ is analytic at 0 with dominant singularity z_0, then the coefficient $f_n = [z^n] f(z)$ satisfies*

$$f_n = \theta(n) |z_0|^{-n} \quad \textit{with} \quad \limsup_{n\to\infty} |\theta(n)|^{1/n} = 1.$$

Proof. From Theorem 1.27 we know that the dominant singularity has to lie on the radius of convergence, and in addition $R := |z_0|$ is the radius of convergence of $f(z)$. Therefore, for small $\varepsilon > 0$, $f_n (R-\varepsilon)^n \to 0$ and $|f_n (R+\varepsilon)^n| \to \infty$. Hence $|f_n| (R-\varepsilon)^n < 1$ for sufficiently large n and so $|f_n| < (R-\varepsilon)^{-n}$ almost everywhere. On the other hand, $|f_n| > (R+\varepsilon)^{-n}$ infinitely often. □

Notation. For two serieses f_n and g_n we write $f_n \sim g_n$ if $\lim_{n\to\infty} \frac{f_n}{g_n} = 1$.

1.3.1 Basic scale

A function that will occur quite often when applying singularity analysis is the function $f(z) = (1-z)^{-\alpha}$, $\alpha \in \mathbb{R}$, which has a dominant singularity at $z = 1$ for $\alpha \notin \{0, -1, -2, \ldots\}$. By Theorem 1.28 the coefficients of the power series expansion of $f(z)$ around 0 have exponential growth 1. But when we are interested in the subexponential factor $\theta(n)$ as well, one has to look more carefully on the type of the singularity $z = 1$, for example the order of the pole 1. Recall Cauchy's coefficient formula from Corollary 1.25. With a smart choice of the loop γ, one can use Cauchy's formula to obtain a more detailed asymptotic estimate of the coefficients f_n.

Theorem 1.29 (expansion for the basic scale [16, Thm. VI.1])**.** *Let*

$$f(z) = (1-z)^{-\alpha}, \quad \alpha \notin \{0, -1, -2, \ldots\}.$$

Then the coefficient $f_n = [z^n] f(z)$ admits for large n a full asymptotic expansion in descending powers of n,

$$f_n = \binom{n+\alpha-1}{n} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \sum_{k=1}^{\infty} \frac{e_k(\alpha)}{n^k}\right),$$

where $\Gamma(\alpha)$ is the Gamma-Function[2] and $e_k(\alpha)$ is a polynomial in α of degree $2k$, $e_1(\alpha) = \frac{1}{2}\alpha(\alpha-1)$, $e_2(\alpha) = \frac{1}{24}\alpha(\alpha-1)(\alpha-2)(3\alpha-1)$, ...

[2] Gamma-Function: $\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt$ for $s \notin \{0, -1, -2, \ldots\}$. It has the property $\Gamma(1) = 1$ and $\Gamma(s+1) = s\Gamma(s)$, so that $\Gamma(n+1) = n!$, $n \in \mathbb{N}$. Further more, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proof. We give here the proof for the first-order term $\frac{n^{\alpha-1}}{\Gamma(\alpha)}$ only (taken from [37]). Using the generalized binomial coefficient,

$$f_n = [z^n](1-z)^{-\alpha} = [z^n]\sum_{k\geq 0}\binom{-\alpha}{k}(-z)^k = \binom{-\alpha}{n}(-1)^n = \binom{n+\alpha-1}{n} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}.$$

By Stirling's Formula $\Gamma(n+1) = n! \sim \left(\frac{n}{e}\right)^n\sqrt{2\pi n}$,

$$\begin{aligned}
\frac{\Gamma(n+\alpha)}{\Gamma(n+1)} &\sim \left(\frac{n+\alpha-1}{e}\right)^{n+\alpha-1}\left(\frac{e}{n}\right)^n\sqrt{\frac{n+\alpha}{n}} \\
&\sim \left(\frac{n+\alpha-1}{n}\right)^n\left(\frac{n+\alpha-1}{e}\right)^{\alpha-1} \\
&\sim (n+\alpha-1)^{\alpha-1} \\
&\sim n^{\alpha-1}.
\end{aligned}$$

The asymptotic expansion of f_n in higher order terms can be computed by a more accurate approximation of $n!$ [16]. □

1.3.2 Implicitly defined functions

Sometimes the function $f(z)$ is not known explicitly so that their singularities cannot be read off from its formulation. When the function $f(z)$ is defined implicitly by a functional equation, singularities are often determined by a point where the implicit function theorem cannot be applied. The following theorem is from [16] and the higher order expansion from [15, 28].

Theorem 1.30 (positive implicit functions [16, Thm. VII.3], [15, 28])**.** *Let $H(x,y)$ be a bivariate function that is analytic in a complex domain $|x| < R$ and $|y| < S$, i.e.,*

$$H(x,y) = \sum_{m,n\geq 0} h_{m,n}x^m y^n, \qquad |x| < R,\ |y| < S.$$

Let $h_{0,0} = H(0,0) = 0$, $h_{0,1} = \frac{\partial}{\partial y}H(0,0) = -1$, the remaining $h_{m,n}$ be nonnegative, and let $h_{m,n} > 0$ for some m and some $n \geq 2$. Assume that there exists two numbers $r \in (0,R)$ and $s \in (0,S)$ such that

$$H(r,s) = 0, \quad \frac{\partial}{\partial y}H(r,s) = 0, \tag{1.15}$$

$\frac{\partial^2}{\partial y^2}H(r,s) \neq 0$ and $\frac{\partial}{\partial x}H(r,s) \neq 0$. Assume further that the equation $H(x, Y(x)) = 0$ admits a solution $Y(x)$ that is analytic at 0, has nonnegative coefficients and is aperiodic (it cannot be put under the form $Y(x) = x^a\tilde{Y}\left(x^d\right)$ with $d \geq 2$ and $\tilde{Y}$ analytic at 0).

Then the power series of $Y(x)$ converges at $x = r$ where it has the asymptotic expansion

$$Y(x) = s + \sum_{i\geq 1} Y_i\left(\sqrt{1-\frac{x}{r}}\right)^i, \qquad \text{with } Y_1 = -\sqrt{\frac{2r\frac{\partial}{\partial x}H(r,s)}{\frac{\partial^2}{\partial y^2}H(r,s)}} \neq 0$$

and computable constants $Y_2, Y_3, \ldots$ (The formulas for Y_2 and Y_3 are given at the end of the proof of this theorem.). Moreover,

$$[x^n]Y(x) = -\frac{Y_1}{2\sqrt{\pi n^3}}r^{-n}\left(1 + O\left(\frac{1}{n}\right)\right).$$

Proof. For a function $L(x,y)$ analytic at (r,s), we let $L_{i,j} := \frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial y^j} L(r,s)$ be the (i,j)-th derivative of $L(x,y)$ at the point (r,s). From

$$H(x,y) = \sum_{i,j\geq 0} \frac{H_{i,j}}{i!j!}(x-r)^i (y-s)^j$$

and $H_{0,0} = H_{0,1} = 0$, $H_{0,2} \neq 0$, $H_{1,0} \neq 0$ it follows that

$$0 = H(x, Y(x)) = H_{1,0}(x-r) + \frac{1}{2}H_{0,2}(Y(x)-s)^2 + O((x-r)(Y(x)-s)) + O\left((Y(x)-s)^3\right).$$

Therefore,

$$(Y(x)-s)^2 \sim -\frac{2H_{1,0}}{H_{0,2}}(x-r) = \frac{2rH_{1,0}}{H_{0,2}}\left(1-\frac{x}{r}\right)$$

and the sign of Y_1 is determined by $Y(x) < s$ for real $x < r$.

To show how to compute the remaining coefficients Y_i, $i \geq 1$, we follow [15, 28]: Due to the equation (1.15), Assumption 4 in [28] is satisfied for $k = 2$. From the Weierstrass preparation theorem (see, for example, [34]) it follows that there exists functions $A(x,y)$ and $P(x,y)$ such that

$$H(x,y) = A(x,y)\,P(x,y)$$

where $A(x,y)$ is analytic at $(x,y) = (r,s)$, $A(r,s) \neq 0$,

$$P(x,y) = (y-s)^2 + p(x)(y-s) + q(x), \tag{1.16}$$

and $p(x)$ and $q(x)$ are analytic with $p(r) = q(r) = 0$. Plotkin and Rosenthal [28] provided formulas to compute the partial derivatives of $A(x,y)$ and $P(x,y)$ from those of $F(x,y)$:

$$P_{i,j} = \begin{cases} 2, & i=0,\ j=2, \\ 0, & i=0,\ j\neq 2, \\ 0, & i\geq 1,\ j=2, \\ 0, & j>2, \\ \dfrac{1}{A_{0,0}}\left(H_{i,j} - \displaystyle\sum_{p=1}^{i-1}\sum_{q=0}^{j}\binom{i}{p}\binom{j}{q}A_{p,q}P_{i-p,j-q} - \sum_{q=1}^{j}\binom{j}{q}A_{0,q}P_{i,j-q}\right) & \text{otherwise,} \end{cases}$$

and

$$A_{i,j} = \frac{1}{(j+1)(j+2)}\left(H_{i,j+2} - \sum_{p=0}^{i-1}\sum_{q=0}^{j+2}\binom{i}{p}\binom{j+2}{q}A_{p,q}P_{i-p,j-q+2}\right).$$

Now we write $p(x)$ and $q(x)$ as

$$p(x) = \sum_{k\geq 1} p_k\left(1-\frac{x}{r}\right)^k, \quad q(x) = \sum_{k\geq 1} q_k\left(1-\frac{x}{r}\right)^k,$$

where the coefficients can be computed from the $P_{i,j}$ by

$$p_k = \frac{P_{k,1}(-r)^k}{k!}, \quad q_k = \frac{P_{k,0}(-r)^k}{k!}.$$

Next, the expansion of $Y(x) - s$ has the form

$$Y(x) - s = \sum_{l\geq 0} b_l\left(1-\frac{x}{r}\right)^{l+\frac{1}{2}} + \sum_{l\geq 1} c_l\left(1-\frac{x}{r}\right)^l,$$

and from the identity $P(x, Y(x)) = 0$ for x close to r and $x \le r$, it follows by equation (1.16) that

$$0 = (Y(x) - s)^2 + p(x)(Y(x) - s) + q(x).$$

By comparison of coefficients, and using the initial $b_0 = -\sqrt{\frac{2rH_{1,0}}{H_{0,2}}}$, the coefficients b_i and c_i can be computed [28, 15]: $c_l = -\frac{1}{2}p_l$ and

$$b_l = \frac{1}{2b_0}\left(-\sum_{k=1}^{l-1} b_k b_{l-k} + \frac{1}{4}\sum_{k=1}^{l} p_k p_{l-k+1} - q_{l+1}\right).$$

For example, we obtain

$$Y_2 = c_1 = r\frac{H_{1,1}}{H_{0,2}} - \frac{rH_{0,3}H_{1,0}}{H_{0,2}^2}$$

and

$$Y_3 = b_1 = \frac{r^2}{Y_1}\left(\frac{5H_{0,3}^2H_{1,0}^2}{18H_{0,2}^4} - \frac{H_{0,3}H_{1,0}H_{1,1}}{H_{0,2}^3} - \frac{H_{0,4}H_{1,0}^2}{6H_{0,2}^3} + \frac{H_{1,1}^2}{2H_{0,2}^2} + \frac{H_{1,0}H_{1,2}}{H_{0,2}^2} - \frac{H_{2,0}}{2H_{0,2}}\right). \quad \square$$

Example (asymptotic number of trees [20, Sec. 9.5])**.** Recall the generating functions (1.5) and (1.12) (pages 14 and 20) for unlabelled rooted trees and unlabelled unrooted trees on n vertices, respectively:

$$\begin{aligned}\hat{T}(x) &= x\exp\left(\sum_{k\ge1}\frac{1}{k}\hat{T}\left(x^k\right)\right),\\ T(x) &= \hat{T}(x) + \frac{1}{2}\left(\hat{T}\left(x^2\right) - \hat{T}(x)^2\right).\end{aligned}$$

Let $L(x)$ be the exponential generation function for labelled trees, $L(x) = \sum_{n\ge1}\frac{1}{n!}l_nx^n$, with $l_n = n^{n-2}$ the number of labelled trees on n vertices [10]. Then $nl_n \le n!\hat{t}_n$ for all $n \ge 0$, and thus $xL'(x) \le \hat{T}(x)$ for all $x \ge 0$. It is known [16, page 369] that $L(x)$ has radius of convergence e^{-1}, and so also $L'(x)$ has radius of convergence e^{-1}. Since $\hat{T}(x)$ dominates $xL'(x)$, $\hat{T}(x)$ must have a radius of convergence $R_{\hat{T}} \le e^{-1}$. On the other hand, the number of trees is bounded from above by the number of plane trees whose ordinary generating function has radius of convergence $\frac{1}{4}$ [16, page 369]. Thus,

$$\frac{1}{4} \le R_{\hat{T}} \le \frac{1}{e}.$$

Since $\hat{T}(x)$ has positive coefficients and $R_{\hat{T}} < 1$, $\hat{T}\left(x^k\right)$, $k \ge 2$, is analytic for $|x| < \sqrt{R_{\hat{T}}}$. Let

$$H(x,y) := x\exp\left(y + \sum_{k\ge2}\frac{1}{k}\hat{T}\left(x^k\right)\right) - y.$$

Then $H(x,y)$ is analytic for $|x| < \sqrt{R_{\hat{T}}}$ and $y \in \mathbb{C}$, $H\left(x, \hat{T}(x)\right) = 0$, $H(0,0) = 0$, and $\frac{\partial}{\partial y}H(0,0) = -1$. Let $r \in \left(0, \sqrt{R_{\hat{T}}}\right)$ and s be two numbers such that the characteristic equations (1.15) are fulfilled. From the definition of $H(x,y)$ we see $\frac{\partial}{\partial y}H(x,y) = H(x,y) + y - 1$, $\frac{\partial^2}{\partial y^2}H(x,y) = H(x,y) + y$, and $\frac{\partial}{\partial x}H(x,y) = \frac{1}{x}(H(x,y) + y) + (H(x,y) + y)\sum_{k\ge2}\hat{T}'\left(x^k\right)x^{k-1}$. Using (1.15), this implies $s = 1$, $H_{0,2} = s = 1$, and $H_{1,0} = \frac{s}{r} + s\sum_{k\ge2}\hat{T}'\left(r^k\right)r^{k-1} \ne 0$. Applying Theorem 1.30 gives now the expansion

$$\hat{T}(x) = s + \sum_{i\ge1}\hat{T}_i\sqrt{1 - \frac{x}{r}}, \qquad \hat{T}_1 = -\sqrt{2rH_{1,0}}.$$

From above we know $r = R_{\hat{T}}$ and $s = 1$. Therefore,

$$\hat{T}(x) = 1 + \hat{T}_1 X + O\left(X^2\right), \tag{1.17}$$

with $\hat{T}_1 = -\sqrt{2R_{\hat{T}}\frac{\partial}{\partial x}H\left(R_{\hat{T}}, 1\right)}$ and $X = \sqrt{1 - \frac{x}{R_{\hat{T}}}}$. Furthermore,

$$\hat{t}_n \sim -\frac{\hat{T}_1}{2\sqrt{\pi}} n^{-\frac{3}{2}} R_{\hat{T}}^{-n}.$$

An exact solution of (1.15) is not known, but it can be estimated by substitution of $\hat{T}(x)$ by a truncated series $\hat{T}^{[m]}(x) = \sum_{n=1}^{m} \hat{t}_n x^n$ where the coefficients $\hat{t}_n$, $n = 1, \ldots, m$, where computed by some other formula (cf. [20, Ch. 3], for example), and by chopping off the infinite series in $H(x, y)$ at the same index. Therefore, one has to determine r such that

$$r \exp\left(1 + \sum_{k=2}^{m} \frac{1}{k} \hat{T}^{[m]}\left(r^k\right)\right) = 1.$$

This gives the estimates $R_{\hat{T}} \approx 0.338322$ and $\hat{T}_1 \approx -1.55949$.

The generating function for unrooted trees, $T(x) = \hat{T}(x) + \frac{1}{2}\left(\hat{T}\left(x^2\right) - \hat{T}(x)^2\right)$, shows that the singularity of $T(x)$ is determined by the one of $\hat{T}(x)$, and thus we can expect the same exponential growth rate, $R_T = R_{\hat{T}}$. It is also clear that $t_n \leq \hat{t}_n \leq n t_n$, since the counted objects only differ in the rooting of one vertex. For the subexponential part, we compose a more accurate expansion of $\hat{T}(x)$ with $T(x)$:

$$\begin{aligned} T(x) &= 1 + \hat{T}_1 X + \hat{T}_2 X^2 + \hat{T}_3 X^3 + \frac{1}{2}\hat{T}\left(x^2\right) - \frac{1}{2}\left(1 + \hat{T}_1 X + \hat{T}_2 X^2 + \hat{T}_3 X^3\right)^2 + O\left(X^4\right) \\ &= \frac{1}{2} + \frac{1}{2}\hat{T}\left(x^2\right) - \frac{1}{2}\hat{T}_1^2 X^2 - \hat{T}_1 \hat{T}_2 X^3 + O\left(X^4\right). \end{aligned}$$

Using the method described in Theorem 1.30 and $H_{1,1} = H_{1,0}$, $H_{0,3} = H_{0,2} = 1$, we get $\hat{T}_2 = \frac{1}{3}\hat{T}_1^2$. By Theorem 1.29,

$$t_n \sim -\frac{\hat{T}_1^3}{3\Gamma\left(-\frac{3}{2}\right)} n^{-\frac{5}{2}} R_T^{-n}.$$

Chapter 2

Dissections

As pointed out in Section 1.2.4, a good understanding of the two-connected components (blocks) of a graph is the key tool to derive counting formulas for the considered graphs in general. Since any block of an outerplanar graph has a unique hamiltonian cycle [21] and can therefore be embedded in the plane such that this hamiltonian cycle lies on the outer face, the task of counting blocks of unlabelled outerplanar graphs coincides with the task of counting dissections of a convex polygon. From here on, we assume dissections to be embedded in the plane such that all vertices of the hamiltonian cycle lie on the outer face. Read derived ordinary generating functions for dissections by a decomposition algorithm [29]. Starting with oriented dissections which are rooted at an edge on the outside of the polygon, he derived counting formulas for oriented and non-oriented dissections rooted at edges or faces. Using the fact that the dual graph of a dissection is a tree, the dissimilar characteristic theorem for trees (Theorem 1.17, page 20) allows to obtain ordinary generating functions for unrooted dissections, with or without orientation.

We will extend Read's counting formulas [29] to cycle index sums for oriented and non-oriented dissections, unrooted or rooted at an edge, vertex, or face. The cycle index sum for vertex rooted and unrooted dissections are then used to derive cycle index sums and generating functions for outerplanar graphs in the next chapter. Furthermore, we can derive asymptotic estimates for the number of dissections, using singularity analysis of generating functions.

A dissection can be mapped onto itself by two types of permutations: reflections and rotations. We speak of an *oriented* dissection when an orientation is imposed on the hamiltonian cycle of the dissection. Since reflections reverse orientations, they are excluded from the permutation group of an oriented dissection. By rooting at an edge which lies on the outer face, rotations are excluded from the permutation group as well. Thus oriented dissections which are rooted at an edge on the outer face become easy to count and are our starting point for the counting of different types of dissections. Having the cycle index sum for oriented and non-oriented dissections which are rooted an edge on the outer face, we can derive the cycle index sums for dissections rooted at an edge not on the outer face, and for dissections rooted at a face by composition. Finally, unrooted dissections can be counted by consideration of the dual tree of a dissection, by what the cycle index sum for unrooted dissections can be expressed as a combination of the cycle index sums for several types of edge rooted and face rooted dissections.

We will not include the single vertex to any set of dissections. However, we include the single edge to the sets of edge-rooted, vertex-rooted and unrooted dissections for convenience.

Notation (outer-edge, inner-edge, symmetry edge, reflective)**.** We will say that a dissection is *rooted at an outer-edge* when it is rooted at an edge that lies on the outer face. In the same way we say that a dissection is *rooted at an inner-edge* when it is rooted at an edge that does not lie on the outer face. A *symmetry edge* is an inner-edge such that there exists a reflection which fixes this edge. It is clear that a dissection can have at most one symmetry edge. The requirement that it has to lie inside the dissection is only for convenience. An edge rooted dissection is called *reflective* if it allows for a non-trivial reflection.

As seen from the preceding discussion, we will need counting formulas for a complete "zoo" of dissections. The corresponding notation is introduced in Table 2.1. Sets of graphs are notated by calligraphic letters, ordinary generating functions by capitals and counts by small letters.

Notation		Type of dissection	
$\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}$	$E_{\mathrm{o}}^{\mathrm{o}}(x)$	oriented	rooted at an outer-edge
$\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}$	$E_{\mathrm{o}}^{\mathrm{r}}(x)$	reflective	rooted at an outer-edge
$\mathcal{E}_{\mathrm{i}}$	$E_{\mathrm{i}}(x)$	not oriented	rooted at an inner-edge
$\mathcal{E}_{\mathrm{s}}$	$E_{\mathrm{s}}(x)$	not oriented	rooted at a symmetry-edge
$\mathcal{F}^{\mathrm{o}}$	$F^{\mathrm{o}}(x)$	oriented	rooted at an inner face
$\mathcal{F}$	$F(x)$	not oriented	rooted at an inner face
$\mathcal{F}_{\mathrm{s}}$	$F_{\mathrm{s}}(x)$	not oriented	rooted at an inner face which is incident to a symmetry edge
$\mathcal{V}$	$V(x)$	not oriented	rooted at a vertex
$\mathcal{D}$	$D(x)$	not oriented	unrooted

Table 2.1: Notation for several types of dissections

We should start now.

2.1 Rooted at an edge

As pointed out before, oriented outer-edge rooted dissections have a trivial permutation group. Therefore, its number is closely related to the bracketing numbers [31, A001003].

Theorem 2.1 (oriented outer-edge rooted dissections)**.** *The cycle index sum for oriented outer-edge rooted dissections is given by*

$$Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\right)=\frac{s_1}{4}\left(s_1+1-\sqrt{s_1^2-6s_1+1}\right).$$

We prove this theorem to give a demonstration how the Composition theorem 1.11 (page 15) can be applied.

Proof. Consider an oriented edge rooted $k+1$-gon, $k\geq 2$, and label the vertices along the orientation, starting with 1 for the right vertex of the root-edge. We let G be the graph obtained by plugging k graphs from $\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}$ in the edges of the $k+1$-gon other than the root edge, mapping the left vertex of the root edge of the k-th graph with the vertex with label k in the $k+1$-gon, see Figure 2.1.

To apply the Composition theorem, let $X=\{1,\ldots,k\}$, $Y=\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}$. Let A be the identity-group on X, that is, we allow only a trivial permutation along the polygon. For each graph $i\in\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}$, denote by Y_i the vertices of the graph i and by B_i the automorphism-group of the graph i with object set $\tilde{Y}_i$ where $\tilde{Y}_i$ is the set Y_i except of the left vertex of the root edge (to avoid double counting of the vertices which overlap between two neighbouring dissections in the composition), $Z(B_i)=\frac{1}{s_1}Z(\Gamma(i))$. Then the cycle index sum for outer-edge rooted oriented dissections where the root-edge is contained in a face with $k+1$ vertices is obtained by the Composition theorem 1.11,

$$s_1\,Z(A)\left[\sum_{i\in Y}Z(B_i)\right],$$

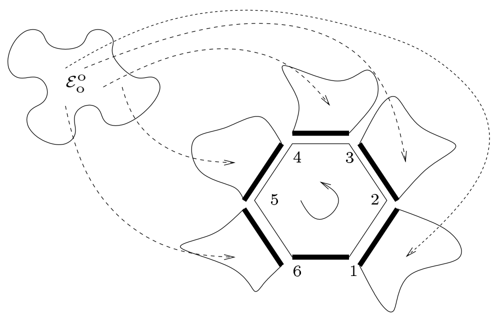

Figure 2.1: Construction of an oriented outer-edge rooted dissection

where the additional factor s_1 is due to the root edge of the polygon. Since $\sum_{i\in\mathcal{E}_o^o} Z(B_i) = \frac{1}{s_1} Z(\mathcal{E}_o^o)$, we obtain

$$s_1\, Z(A)\left[\sum_{i\in Y} Z(B_i)\right] = s_1\, s_1^k \left[\frac{1}{s_1} Z(\mathcal{E}_o^o)\right] = s_1 \left(\frac{Z(\mathcal{E}_o^o)}{s_1}\right)^k .$$

The case of a single edge ($k=1$) contributes s_1^2 to $Z(\mathcal{E}_o^o)$. Thus summing over k yields

$$Z(\mathcal{E}_o^o) = s_1^2 + s_1 \sum_{k\geq 2}\left(\frac{Z(\mathcal{E}_o^o)}{s_1}\right)^k = s_1^2 + s_1 \frac{\left(Z(\mathcal{E}_o^o)/s_1\right)^2}{1 - Z(\mathcal{E}_o^o)/s_1},$$

that is,

$$Z(\mathcal{E}_o^o) = s_1^2 + \frac{Z(\mathcal{E}_o^o)^2}{s_1 - Z(\mathcal{E}_o^o)} \tag{2.1}$$

This equation is quadratic in $Z(\mathcal{E}_o^o)$, and the correct root can be found from the observation that $\frac{Z(\mathcal{E}_o^o)}{s_1}|_{s_1=0} = 0$. □

Corollary 2.2. *The ordinary generating function for oriented outer-edge rooted dissections is given by*

$$E_o^o(x) = \frac{x}{4}\left(x + 1 - \sqrt{x^2 - 6x + 1}\right),$$

the coefficients are $(e_o^o)_2 = 1$,

$$(e_o^o)_n = [x^n]\, E_o^o(x) = \frac{1}{n-1}\sum_{k=0}^{n-3}\binom{2n-k-4}{n-2}\binom{n-3}{k}, \quad n \geq 3,$$

and have the asymptotic estimate

$$(e_o^o)_n = \left(\frac{3}{4} - \frac{1}{\sqrt{2}}\right)\sqrt{\frac{3\sqrt{2}-4}{\pi}}\, n^{-\frac{3}{2}}\, \delta^{-n}\left(1 + O\left(\frac{1}{n}\right)\right) \tag{2.2}$$

with $\delta = 3 - 2\sqrt{2}$.

Proof. The ordinary generating function is given due to Pólya's theorem 1.9 by

$$E_o^o(x) = Z(\mathcal{E}_o^o; x).$$

The formula for the coefficients is taken from [31, A001003], but could also be derived by Lagrange's Inversion Formula [20, Sec. 1.7] applied to the functional equation (2.1).

For the asymptotic estimate of $(e_{\mathrm{o}}^{\mathrm{o}})_n$, we rewrite $E_{\mathrm{o}}^{\mathrm{o}}(x)$ as

$$E_{\mathrm{o}}^{\mathrm{o}}(x) = \frac{x}{4}(x+1) - \frac{x}{4}\sqrt{1-\left(3-2\sqrt{2}\right)x}\,\sqrt{1-\frac{x}{3-2\sqrt{2}}}.$$

Note that $1-\left(3-2\sqrt{2}\right)x > 0$ and $1-\frac{x}{3-2\sqrt{2}} > 0$ for $|x| < 3-2\sqrt{2}$. Hence, $E_{\mathrm{o}}^{\mathrm{o}}(x)$ is analytic at 0 with radius of convergence $\delta = 3-2\sqrt{2}$ and the term $-\frac{x}{4}\sqrt{1-\delta x}$ is analytic for $|x| < 3+2\sqrt{2}$. Therefore, it can be expanded around $x=\delta$,

$$-\frac{x}{4}\sqrt{1-\delta x} = \sum_{m\geq 0} a_m (\delta - x)^m$$

with $a_m = -\frac{1}{4}\frac{1}{m!}\frac{\partial^m}{\partial x^m} x\sqrt{1-\delta x}\,|_{x=\delta}$, especially $a_0 = -\frac{\delta}{4}\sqrt{1-\delta^2}$. Hence,

$$\begin{aligned} E_{\mathrm{o}}^{\mathrm{o}}(x) &= \frac{x}{4}(x+1) + \sum_{m\geq 0} a_m (\delta - x)^m \sqrt{1-\frac{x}{\delta}} \\ &= \frac{x}{4}(x+1) + \sum_{m\geq 0} a_m \delta^m \left(1-\frac{x}{\delta}\right)^{m+\frac{1}{2}} \end{aligned}$$

From Theorem 1.29 (with $\alpha = -\left(m+\frac{1}{2}\right)$) it follows that

$$\begin{aligned} [x^n]\sqrt{1-\frac{x}{\delta}} &\sim \frac{n^{-\frac{3}{2}}}{\Gamma\left(-\frac{1}{2}\right)}\delta^{-n}, \\ [x^n]\left(1-\frac{x}{\delta}\right)^{m+\frac{1}{2}} &= o\left(n^{-\frac{3}{2}}\delta^{-n}\right), \quad m \geq 1, \end{aligned}$$

and $a_m = o\left(\delta^{-m}\right)$. Therefore,

$$\begin{aligned} (e_{\mathrm{o}}^{\mathrm{o}})_n = [x^n]\,E_{\mathrm{o}}^{\mathrm{o}}(x) &\sim \frac{a_0}{\Gamma\left(-\frac{1}{2}\right)} n^{-\frac{3}{2}}\delta^{-n} \\ &= \left(\frac{3}{4}-\frac{1}{\sqrt{2}}\right)\sqrt{\frac{3\sqrt{2}-4}{\pi}}\,n^{-\frac{3}{2}}\left(3+2\sqrt{2}\right)^n \\ &\approx 0.0119205\, n^{-\frac{3}{2}}\, 5.82843^n. \end{aligned}$$

□

A non-oriented outer-edge rooted dissection is counted twice by $Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\right)$ when the dissection is not invariant under the reflection that fixes the root-edge. Therefore, to derive counting formulas for non-oriented dissections later, we need the cycle index sums for reflective outer-edge rooted dissections. The set of automorphisms can here be divided into two classes. The class which members are the identity mappings and the class of permutations which apply the reflection. We count them separately.

Theorem 2.3 (reflective outer-edge rooted dissections)**.** *The cycle index sum for reflective outer-edge rooted dissections is given by*

$$Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right) = \frac{1}{2}\left(Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right) + Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)\right),$$

where $Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)$ *counts only the identity mappings,*

$$Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right) := 2Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)|_{s_2=0} = \frac{1+s_1-3s_1^2+s_1^3-(1+s_1)\sqrt{s_1^4-6s_1^2+1}}{4s_1}$$

and $Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)$ *counts the reflection mappings,*

$$Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right) := 2Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right) - Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right) = \frac{s_2^2-3s_1s_2+s_2+s_1-(s_2+s_1)\sqrt{s_2^2-6s_2+1}}{4s_2}.$$

Proof. Similarly to the proof of Theorem 2.1 we consider an edge rooted polygon on $k+1$, $k \geq 2$, vertices. When $k = 2m$ is even, $m \geq 1$, we plug oriented edge rooted dissections in the edges of the polygon other than the root edge such that edges which have the same distance from the root edge of the polygon receive the same dissection, see Figure 2.2. The identity mapping is counted by

$$s_1 \left(\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_1^2\right)}{s_1^2} \right)^m$$

and the reflection mapping by

$$s_1 \left(\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2\right)}{s_2} \right)^m,$$

where the additional factor s_1 is due to the vertex opposite to the root edge, which is fixed by both, the identity and the reflection mapping.

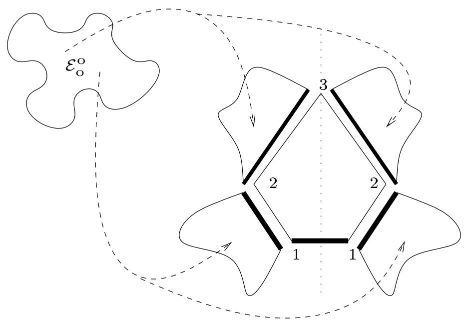

Figure 2.2: Construction of a reflective outer-edge rooted dissection when k is even

When $k = 2m+1$ is odd, $m \geq 1$, we again plug oriented edge rooted dissections in the edges of the polygon other than the root edge such that edges at the same distance from the root edge receive the same dissection, but for the edge opposite to the root edge we have to add a reflective edge rooted dissection now, see Figure 2.3. The identity mapping is now counted by

$$\left(\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_1^2\right)}{s_1^2} \right)^m Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)$$

and the reflection mapping by

$$\left(\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2\right)}{s_2} \right)^m Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right).$$

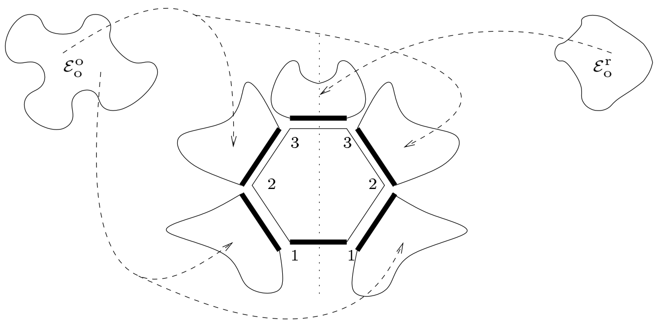

Figure 2.3: Construction of a reflective outer-edge rooted dissection when k is odd.

The single edge contributes s_1^2 for the identity mapping and s_2 for the reflection. Thus summing over $k \geq 1$ yields

$$Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right) = s_1^2 + \sum_{m \geq 1} \left(\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_1^2\right)}{s_1^2} \right)^m \left(s_1 + Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)\right) = s_1^2 + \frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_1^2\right)}{s_1^2 - Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_1^2\right)} \left(s_1 + Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)\right),$$

that is

$$Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right) = \frac{s_1^4 + \left(s_1 - s_1^2\right) Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_1^2\right)}{s_1^2 - 2Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_1^2\right)},$$

and

$$Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right) = s_2 + \sum_{m \geq 1} \left(\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2\right)}{s_2} \right)^m \left(s_1 + Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)\right) = s_2 + \frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2\right)}{s_2 - Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2\right)} \left(s_1 + Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)\right),$$

that is

$$Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right) = \frac{s_2^2 + \left(s_1 - s_2\right) Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2\right)}{s_2 - 2Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2\right)}.$$

Application of Theorem 2.1 yields the statement. □

By application of Pólya's theorem we can transform the cycle index sum to an ordinary generating function, substituting s_i by x^i, $i \geq 1$.

Corollary 2.4. *The ordinary generating function for reflective outer-edge rooted dissections is given by*

$$E_{\mathrm{o}}^{\mathrm{r}}(x) = \frac{1 + x + (x-3)\,x^2 - (1+x)\sqrt{x^4 - 6x^2 + 1}}{4x}.$$

Having the generating functions for oriented and reflective outer-edge rooted dissections in our hand, it is not difficult to derive the generating function for non-oriented outer-edge rooted dissections. But, since we do not need these counting formulas, we omit this step here and fetch it later in a more general setting (Section 3.4). We now turn over to inner-edge rooted dissections instead, which we can construct by joining two outer-edge rooted dissections on their root edge and consideration of the possible reflections and rotations.

Theorem 2.5 (inner-edge rooted dissections)**.** *The cycle index sum for inner-edge rooted dissections is given by*

$$\begin{aligned} Z\left(\mathcal{E}_{\mathrm{i}}\right) &= \frac{\left(3s_1 - 1 + \sqrt{s_1^2 - 6s_1 + 1}\right)^2}{64} + \frac{\left(s_1 + s_2\right)^2}{16 s_2} \left(\frac{1 + s_2 - \sqrt{s_2^2 - 6s_2 + 1}}{1 - s_2 + \sqrt{s_2^2 - 6s_2 + 1}} \right)^2 \\ &\quad - \frac{s_1^2 + s_2}{16 s_2} \left(3s_2 - 1 + \sqrt{s_2^2 - 6s_2 + 1}\right). \end{aligned}$$

Proof. To enumerate dissections rooted at an inner edge, consider the four kinds of transformations of the plane that map the root edge onto itself (check also Figure 2.4):

1. The identity mapping
2. Reflection about the root-edge
3. Rotation about the root-edge
4. Reflection about the perpendicular bisector of the root-edge

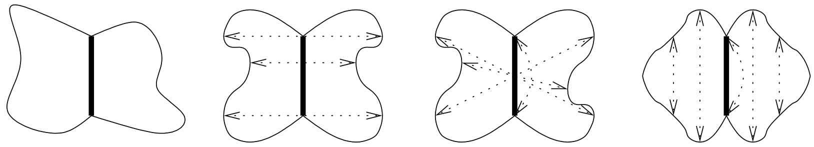

Figure 2.4: The four kinds of transformations that map the root edge of an inner-edge rooted dissection onto itself.

Those dissections which are invariant under the identity mapping are counted by $\left(\frac{Z(\mathcal{E}_\mathrm{o}^\mathrm{o})-s_1^2}{s_1}\right)^2$. Those which are invariant under reflections about the root edge can be constructed by taking a non-empty oriented outer-edge rooted dissection and join it with a (reflected) copy of itself on its root edge where the vertices of the root edge remain fixed. Those which are invariant under rotations about the root edge can be constructed the same way, but the vertices of the root edge have to be interchanged. Therefore, those dissections which are invariant under the second or third mapping are counted by $\frac{Z(\mathcal{E}_\mathrm{o}^\mathrm{o};s_2)-s_2^2}{s_2^2}\left(s_1^2+s_2\right)$. The dissections which are invariant under the fourth kind of mapping are constructed by placing together at the root edge two independently chosen non-empty reflective outer-edge rooted dissections. They are counted by $\left(\frac{Z_\mathrm{r}(\mathcal{E}_\mathrm{o}^\mathrm{r})-s_2}{s_2}\right)^2 s_2$. In sum, we get

$$Z\left(\mathcal{E}_\mathrm{i}\right)=\frac{1}{4}\left(\left(\frac{Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}\right)}{s_1}-s_1\right)^2+\left(\frac{Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o};s_2\right)}{s_2^2}-1\right)\left(s_1^2+s_2\right)+\left(\frac{Z_\mathrm{r}\left(\mathcal{E}_\mathrm{o}^\mathrm{r}\right)}{s_2}-1\right)^2 s_2\right),$$

which together with Theorem 2.1 and Theorem 2.3 yields the statement. □

Corollary 2.6. *The ordinary generating function for inner-edge rooted dissections is given by*

$$\begin{aligned} E_\mathrm{i}\left(x\right) &= \frac{\left(3x-1+\sqrt{x^2-6x+1}\right)^2}{64}+\frac{(1+x)^2}{16}\left(\frac{1+x^2-\sqrt{x^4-6x^2+1}}{1-x^2+\sqrt{x^4-6x^2+1}}\right)^2 \\ &\quad -\frac{3x^2-1+\sqrt{x^4-6x^2+1}}{8}. \end{aligned}$$

Theorem 2.7 (symmetry-edge rooted dissections)**.** *The cycle index sum for symmetry-edge rooted dissections is given by*

$$\begin{aligned} Z\left(\mathcal{E}_\mathrm{s}\right) &= \frac{s_1^6-2s_1^6s_2^2+s_1^4s_2^2-s_1^2s_2^3-s_2^3}{16s_1^4s_2^3}+\frac{3}{8}\left(1-s_2-s_1^2\right) \\ &\quad +\frac{1+s_1^2}{16s_1^4}\sqrt{s_1^8-6s_1^4+1}-\frac{1}{8}\sqrt{s_1^4-6s_1^2+1} \\ &\quad -\frac{1}{16s_2}\left(1+\frac{s_1^2}{s_2^2}\right)\sqrt{s_2^4-6s_2^2+1}-\frac{1}{16}\left(1+\frac{s_1^2}{s_2}\right)\sqrt{s_2^2-6s_2+1}. \end{aligned}$$

Proof. Non-empty dissections rooted at a symmetry edge can be constructed by taking an oriented outer-edge rooted dissection and joining it at the root-edge with a copy of itself. For dissections which are not invariant under reflection, this can be done in two ways. Thus the permutations which fix all vertices have to be counted twice, while the ones which express the reflection about the symmetry edge divide in those which fix the root vertices and those which permute them (see Figure 2.4, second and third picture). Hence, there are counted by

$$\frac{1}{4}\left(2\,\frac{Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o};s_1^2\right)-Z_\mathrm{id}\left(\mathcal{E}_\mathrm{o}^\mathrm{r};s_1^2,s_2^2\right)}{s_1^4}\,s_1^2+\frac{Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o};s_2\right)-Z_\mathrm{id}\left(\mathcal{E}_\mathrm{o}^\mathrm{r};s_2,s_4\right)}{s_2^2}\left(s_1^2+s_2\right)\right).$$

For reflection-invariant dissections, we distinguish again between those permutations which keep the vertices of the root-edge fixed (Figure 2.4, first and second picture),

$$s_1^2 \frac{Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2\right) - s_1^4}{s_1^4} \text{ and } s_1^2 \frac{Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_2, s_4\right) - s_2^2}{s_2^2},$$

and those permutations which interchange the root-vertices (see Figure 2.4, third and fourth picture),

$$s_2 \frac{Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2\right) - s_2^2}{s_2^2} \text{ and } s_2 \frac{Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_2, s_4\right) - s_2^2}{s_2^2}.$$

So, we get

$$\begin{aligned} Z\left(\mathcal{E}_{\mathrm{s}}\right) &= \frac{1}{4}\left(\frac{2}{s_1^2}\left(Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_1^2\right) - Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2\right)\right) + \frac{s_1^2 + s_2}{s_2^2}\left(Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2\right) - Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_2, s_4\right)\right)\right) \\ &+ \frac{1}{4}\left(\frac{1}{s_1^2} Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2\right) - s_1^2 + \frac{s_1^2}{s_2^2} Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_2, s_4\right) - s_1^2\right) \\ &+ \frac{1}{4}\left(\frac{1}{s_2} Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2\right) - s_2 + \frac{1}{s_2} Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_2, s_4\right) - s_2\right), \end{aligned}$$

which is simplified to

$$Z\left(\mathcal{E}_{\mathrm{s}}\right) = \frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_1^2\right)}{2 s_1^2} + \frac{s_1^2 + s_2}{4 s_2^2} Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2\right) - \frac{Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2\right)}{4 s_1^2} + \frac{Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2\right)}{4 s_2} - \frac{s_1^2 + s_2}{2}. \tag{2.3}$$

This together with Theorem 2.1 and Theorem 2.3 yields the statement. □

When Pólya's Theorem is applied to equation (2.3), the terms for the reflective dissections cancel out and we obtain a simple formula for the generating function of symmetry-edge rooted dissections.

Corollary 2.8. *The ordinary generating function for symmetry-edge rooted dissections is given by*

$$E_{\mathrm{s}}(x) = \frac{E_{\mathrm{o}}^{\mathrm{o}}\left(x^2\right)}{x^2} - x^2 = \frac{1}{4}\left(1 - 3x^2 - \sqrt{x^4 - 6x^2 + 1}\right).$$

2.2 Rooted at a face

When a dissection is rooted at a face, its automorphism group might include cyclic permutations. Therefore we have to deal with cyclic groups now. We start with the oriented case again, since it forbids possible reflections.

Theorem 2.9 (oriented face rooted dissections)**.** *The cycle index sum for oriented face rooted dissections is given by*

$$\begin{aligned} Z\left(\mathcal{F}^{\mathrm{o}}\right) &= -\sum_d \frac{\varphi(d)}{d} \log\left(\frac{3}{4} - \frac{1}{4} s_d + \frac{1}{4}\sqrt{s_d^2 - 6 s_d + 1}\right) \\ &- \frac{s_1^2 + 2 s_1 + 2 s_2 + 7}{16} + \frac{s_1 + 5}{16}\sqrt{s_1^2 - 6 s_1 + 1} + \frac{1}{8}\sqrt{s_2^2 - 6 s_2 + 1}. \end{aligned}$$

Proof. Consider an oriented unrooted k-gon, $k \geq 3$, with the cycle group C_k. An oriented face rooted dissection can be constructed by taking k oriented outer-edge rooted dissections and plugging them in the edges of the polygon, see Figure 2.5. By the Composition theorem 1.11 it follows that the cycle index sum for oriented dissections rooted at a face of size k is given by

$$Z\left(C_k\right)\left[\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\right)}{s_1}\right].$$

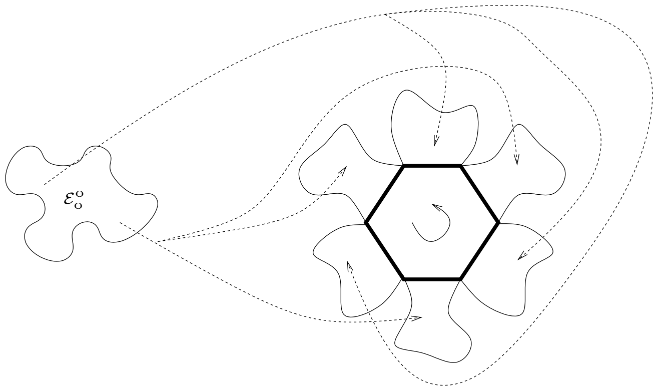

Figure 2.5: Construction of an oriented face rooted dissection counted by $s_3^2\left[\frac{Z(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}})}{s_1}\right]$

Summing over k yields

$$\begin{aligned} Z\left(\mathcal{F}^{\mathrm{o}}\right) &= \sum_{k\geq 3} Z\left(C_k\right)\left[\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\right)}{s_1}\right] \\ &= \sum_{k\geq 1} Z\left(C_k\right)\left[\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\right)}{s_1}\right] - Z\left(C_1\right)\left[\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\right)}{s_1}\right] - Z\left(C_2\right)\left[\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\right)}{s_1}\right]. \end{aligned}$$

Recall formula (1.3) for the cycle index of the cyclic group,

$$Z\left(C_k\right) = \frac{1}{k}\sum_{d|k}\varphi\left(d\right)s_d^{k/d}.$$

Then $Z\left(C_1\right) = s_1$, $Z\left(C_2\right) = \frac{1}{2}\left(s_1^2 + s_2\right)$, and

$$\sum_{k\geq 1} Z\left(C_k\right) = \sum_{k\geq 1}\frac{1}{k}\sum_{d|k}\varphi\left(d\right)s_d^{k/d} = \sum_{d,m}\frac{\varphi\left(d\right)}{dm}s_d^m = \sum_{d} -\frac{\varphi\left(d\right)}{d}\log\left(1-s_d\right),$$

and thus

$$Z\left(\mathcal{F}^{\mathrm{o}}\right) = -\left\{\sum_{d}\frac{\varphi\left(d\right)}{d}\log\left(1-\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}};s_d\right)}{s_d}\right)\right\} - \frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\right)}{s_1} - \frac{1}{2}\left(\left(\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\right)}{s_1}\right)^2 + \frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}};s_2\right)}{s_2}\right),$$

which together with Theorem 2.1 yields the statement. □

Corollary 2.10. *The ordinary generating function for oriented face rooted dissections is given by*

$$\begin{aligned} F^{\mathrm{o}}\left(x\right) &= -\sum_{d}\frac{\varphi\left(d\right)}{d}\log\left(\frac{3}{4}-\frac{1}{4}x^d+\frac{1}{4}\sqrt{x^{2d}-6x^d+1}\right) - \frac{3x^2+2x+7}{16} \\ &\quad + \frac{x+5}{16}\sqrt{x^2-6x+1} + \frac{1}{8}\sqrt{x^4-6x^2+1}. \end{aligned}$$

When we drop the requested orientation, we have to deal with dihedral groups instead of cyclic groups.

Theorem 2.11 (face rooted dissections)**.** *The cycle index sum for face rooted dissections is given by*

$$
\begin{aligned}
Z(\mathcal{F}) = &-\frac{1}{2}\sum_d \frac{\varphi(d)}{d}\log\left(\frac{3}{4}-\frac{1}{4}s_d+\frac{1}{4}\sqrt{s_d^2-6s_d+1}\right)+\frac{s_1+5}{32}\sqrt{s_1^2-6s_1+1}\\
&-\frac{s_1^2+2s_1+2s_2+7}{32}+\frac{s_1^2+6s_1s_2+8s_1^2s_2+3s_2^2+4s_1s_2^2+5s_1^2s_2^2+2s_2^3-2s_1s_2^3-2s_1^2s_2^3-s_2^4}{16s_2\left(1-s_2+\sqrt{s_2^2-6s_2+1}\right)^2}\\
&+\left(\frac{1}{16}-\frac{s_1^2+2s_1s_2+3s_1^2s_2+s_2^2-2s_1s_2^2-2s_1^2s_2^2-s_2^3}{8s_2\left(1-s_2+\sqrt{s_2^2-6s_2+1}\right)^2}\right)\sqrt{s_2^2-6s_2+1}.
\end{aligned}
$$

Proof. Consider an non-oriented unrooted k-gon, $k \geq 3$, with the dihedral group D_k, generated by the cycle $(1\,2\,\cdots\,k)$ and the reflection $(1\,k)\,(2\,k-1)\cdots$. Recall formula (1.4) for the cycle index of the dihedral group,

$$
Z(D_k)=\frac{1}{2}Z(C_k)+\begin{cases}\frac{1}{2}s_1s_2^m, & k=2m+1 \text{ odd},\\ \frac{1}{4}s_2^{m+1}+\frac{1}{4}s_1^2s_2^m, & k=2m+2 \text{ even}.\end{cases}
$$

Again we construct a face rooted dissection by plugging oriented outer-edge rooted dissections in the edges of the polygon. Those permutations which correspond to cycles are counted by $Z(C_k)$. For those permutations which correspond to reflections, the outer-edge rooted dissections attached to corresponding pairs of edges must be the same, while the outer-edge rooted dissection which is mapped to itself must be reflective. There are three cases to distinguish.

For $k = 2m+1$ odd, the term $s_1s_2^m$ in $Z(D_k)$ corresponds to m oriented outer-edge rooted dissections which are attached to corresponding pairs of edges, and one outer-edge rooted reflective dissection, see Figure 2.6. So, $s_1s_2^m$ has to be replaced by

$$
s_1\frac{Z_{\mathrm{r}}(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}})}{s_2}\left(\frac{Z(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}};s_2)}{s_2}\right)^m
$$

where the extra s_1 is due to the vertex of the root-face which is fixed by the reflection.

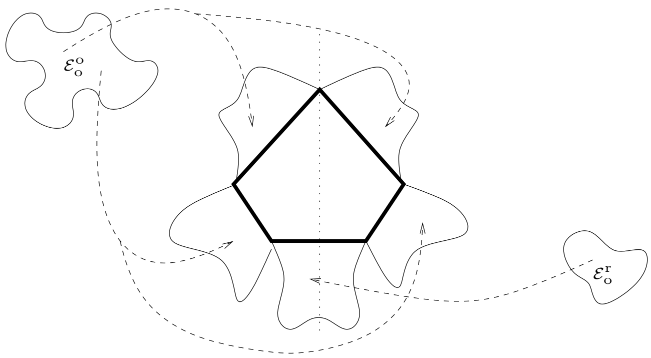

Figure 2.6: The extra term $s_1s_2^m$ in $Z(D_k)$ for odd k (here: $m = 2$).

For $k = 2m+2$ even, the term s_2^{m+1} corresponds to $m+1$ oriented outer-edge rooted dissections which are attached to corresponding pairs of edges (Figure 2.7, left), and thus s_2^{m+1} has to be replaced by

$$
\frac{s_1^2}{s_2}\left(\frac{Z(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}};s_2)}{s_2}\right)^{m+1}
$$

where the term $\frac{s_1^2}{s_2}$ is due to the vertices of the root-face which are fixed by the reflection.

Finally, the term $s_1^2 s_2^m$ corresponds to m oriented outer-edge rooted dissections which are attached to corresponding pairs of edges, and two outer-edge rooted reflective dissections (Figure 2.7, right), and thus it has to be replaced by

$$\frac{Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)^2}{s_2}\left(\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2\right)}{s_2}\right)^m.$$

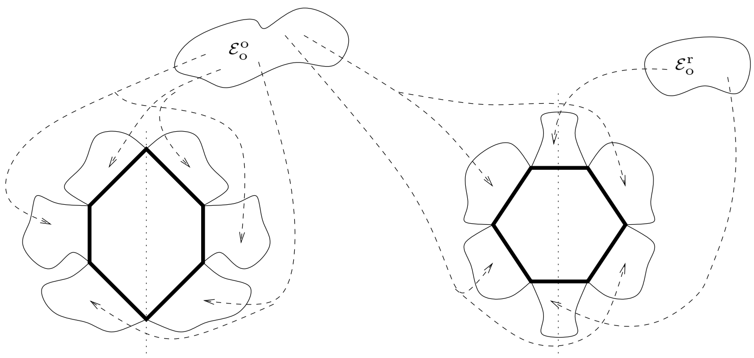

Figure 2.7: The extra terms $s_2^{m+1}$and $s_1^2 s_2^m$ in $Z\left(D_k\right)$ for even k (here: $m=2$).

Hence,

$$\begin{aligned}
Z\left(\mathcal{F}\right) &= \frac{1}{2}\sum_{k\geq 3} Z\left(C_k\right)\left[\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\right)}{s_1}\right] + \frac{1}{2}\sum_{m\geq 1}\left(\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2\right)}{s_2}\right)^m\left(\frac{s_1}{s_2} Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right) + \frac{s_1^2}{2s_2^2} Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2\right) + \frac{Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)^2}{2s_2}\right)\\
&= \frac{1}{2} Z\left(\mathcal{F}^{\mathrm{o}}\right) + \frac{1}{2s_2}\left(s_1 Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right) + \frac{s_1^2}{2s_2} Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2\right) + \frac{1}{2} Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)^2\right)\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2\right)}{s_2 - Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2\right)},
\end{aligned}$$

which together with the Theorems 2.1, 2.3, and 2.9 yields the statement. □

Corollary 2.12. *The ordinary generating function for face rooted dissections is given by*

$$\begin{aligned}
F\left(x\right) = -\frac{1}{2}\sum_{d}\frac{\varphi\left(d\right)}{d}\log\left(\frac{3}{4} - \frac{1}{4}x^d + \frac{1}{4}\sqrt{x^{2d} - 6x^d + 1}\right) - \frac{3x^2 + 2x + 7}{32}\\
+ \frac{3x^4 + 2x^3 - 4x^2 - 2x - 1}{8\left(1 - x^2 + \sqrt{x^4 - 6x^2 + 1}\right)^2} + \frac{1 + 6x + 11x^2 + 4x^3 + 7x^4 - 2x^5 - 3x^6}{16\left(1 - x^2 + \sqrt{x^4 - 6x^2 + 1}\right)^2}\\
+ \frac{x + 5}{32}\sqrt{x^2 - 6x + 1} + \frac{1}{16}\sqrt{x^4 - 6x^2 + 1}.
\end{aligned}$$

The next type of dissections can be identified with those which are rooted at a symmetry edge. Hence, their generating functions agree, $F_{\mathrm{s}}\left(x\right) = E_{\mathrm{s}}\left(x\right)$. But since we change the rooting a bit, the permutation groups and the cycle index sums differ.

Theorem 2.13. *The cycle index sum for face rooted dissections where the root-face is incident to a symmetry edge is given by*

$$\begin{aligned}
Z\left(\mathcal{F}_{\mathrm{s}}\right) = \frac{s_1^6\left(1 - 3s_2^2 - 3s_2^3\right) + s_1^4\left(s_2^2 + 5s_2^3 - 3s_2^4\right) - s_2^3 - s_1^2 s_2^3}{8s_1^4 s_2^3}\\
- \frac{1}{4}\sqrt{s_1^4 - 6s_1^2 + 1} + \frac{s_2^3 + s_1^2 s_2^3}{8s_1^4 s_2^3}\sqrt{s_1^8 - 6s_1^4 + 1} - \frac{s_1^2 + s_2^2}{8s_2^3}\sqrt{s_2^4 - 6s_2^2 + 1}.
\end{aligned}$$

Proof. As for symmetry-edge rooted dissections we join two nonempty oriented outer-edge rooted dissections on their root-edge and declare one of the faces incident to the root edge to be the root face. For outer-edge rooted dissections not invariant under reflection (see Figure 2.4, first picture) we obtain the term

$$\frac{1}{s_1^2}\left(Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_1^2\right) - Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2\right)\right).$$

For reflective outer-edge rooted dissections, we have to take care of the additional reflection that can occur here (see Figure 2.4, last picture). We obtain

$$\frac{1}{2}\left(\frac{Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2\right) - s_1^4}{s_1^4} s_1^2 + \frac{Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2\right) - s_2^2}{s_2^2} s_2\right).$$

So,

$$\begin{aligned}
Z\left(\mathcal{F}_{\mathrm{s}}\right) &= \frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_1^2\right) - Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2\right)}{s_1^2} + \frac{1}{2}\left(\frac{Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2\right) - s_1^4}{s_1^2} + \frac{Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2\right) - s_2^2}{s_2}\right) \\
&= \frac{1}{s_1^2} Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_1^2\right) - \frac{1}{2 s_1^2} Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2\right) + \frac{1}{2 s_2} Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2\right) - \frac{1}{2}\left(s_1^2 + s_2\right),
\end{aligned}$$

which together with Theorem 2.1 and Theorem 2.3 yields the result. □

2.3 Unrooted dissections

We are now ready to derive the cycle index sum for unrooted non-oriented dissections.

Theorem 2.14 (dissections)**.** *The cycle index sum for dissections is given by*

$$\begin{aligned}
Z(\mathcal{D}) &= -\frac{1}{2}\sum_d \frac{\varphi(d)}{d} \log\left(\frac{3}{4} - \frac{1}{4} s_d + \frac{1}{4}\sqrt{s_d^2 - 6 s_d + 1}\right) + \frac{s_2 + s_1^2 - 4 s_1 - 2}{16} \\
&\quad + \frac{s_1^2 - 3 s_1^2 s_2 + 2 s_1 s_2}{16 s_2^2} + \frac{3 - s_1}{16}\sqrt{s_1^2 - 6 s_1 + 1} - \frac{1}{16}\left(1 + \frac{s_1^2}{s_2^2} + \frac{2 s_1}{s_2}\right)\sqrt{s_2^2 - 6 s_2 + 1}.
\end{aligned}$$

Proof. Observe that the dual of a dissection which has at least one face is a tree. Faces of the dissection correspond thereby to vertices of the dual tree and inner (respectively symmetry) edges of the dissection correspond to inner (respectively symmetry) edges of the tree. A vertex of the tree which is not incident to a symmetry edge corresponds then to a face of the dissection which is not incident to a symmetry edge. We can therefore apply the dissimilarity characteristic theorem 1.18, replacing $\mathcal{V}_1$ by $\mathcal{F} \setminus \mathcal{F}_{\mathrm{s}}$, $\mathcal{E}$ by $\mathcal{E}_{\mathrm{i}}$, and $\mathcal{S}$ by $\mathcal{E}_{\mathrm{s}}$,

$$Z(\mathcal{D}) = \frac{1}{2}\left(s_1^2 + s_2\right) + Z(\mathcal{F}) - Z\left(\mathcal{F}_{\mathrm{s}}\right) - Z\left(\mathcal{E}_{\mathrm{i}}\right) + 2 Z\left(\mathcal{E}_{\mathrm{s}}\right).$$

The additional term $\frac{1}{2}\left(s_1^2 + s_2\right)$ counts the dissection which contains one edge only (the dual of this graph is the empty tree). Application of the Theorems 2.5, 2.7, 2.11, and 2.13 yields the result. □

Corollary 2.15. *The ordinary generating function for dissections is given by*

$$\begin{aligned}
D(x) &= -\frac{1}{2}\sum_d \frac{\varphi(d)}{d} \log\left(\frac{1}{4}\left(3 - x^d + \sqrt{x^{2d} - 6 x^d + 1}\right)\right) \\
&\quad + \frac{x^2}{8} - \frac{1}{4} x - \frac{5}{16} + \frac{1}{8x} + \frac{1}{16 x^2} + \frac{3 - x}{16}\sqrt{x^2 - 6x + 1} - \frac{1 + 2x + x^2}{16 x^2}\sqrt{x^4 - 6 x^2 + 1}
\end{aligned}$$

and its coefficients have the asymptotic estimate

$$d_n \sim \frac{\left(3\sqrt{2}-4\right)^{\frac{3}{2}}}{8\sqrt{2\pi}} n^{-\frac{5}{2}} \left(3+2\sqrt{2}\right)^n .$$

Proof. $D(x)$ is obtained by Pólya's theorem again, $D(x) = Z(\mathcal{D}; x)$. The dominant singularity of $D(x)$ is either determined from a zero in the radical of a square-root-term or the parameter of a logarithm. Let us look at the log-terms in $D(x)$. We have $3 - x + \sqrt{x^2 - 6x + 1} = 0$ for $x = \frac{1}{2}\left(7 \pm \sqrt{33}\right) > \delta$ ($\delta = 3 - 2\sqrt{2}$ as in Corollary 2.2), and thus the dominant singularity is determined by a zero in the radical of a square root. Since $\delta^{1/d} > \delta$ for all $d \geq 2$ the dominant singularity is at $x = \delta$ and determined by terms which include $\sqrt{x^2 - 6x + 1}$. We can therefore rewrite $D(x)$ in the form

$$D(x) = -\frac{1}{2} \log\left(1 - \frac{\sqrt{x^2-6x+1}}{x-3}\right) + \frac{3-x}{16}\sqrt{x^2-6x+1} + A(x),$$

where $A(x)$ is analytic at 0 with radius of convergence $> \delta$. Furthermore, since the logarithm-term is analytic for $|x| < \delta$, we get

$$\begin{aligned}
-\frac{1}{2} \log\left(1 - \frac{\sqrt{x^2-6x+1}}{x-3}\right) &= \frac{1}{2}\sum_{k\geq 1} \frac{1}{k}\left(\frac{\sqrt{x^2-6x+1}}{x-3}\right)^k \\
&= \frac{1}{2(x-3)}\sqrt{x^2-6x+1} + \frac{x^2-6x+1}{4(x-3)^2} \\
&\quad + \frac{1}{6(x-3)^3}\left(\sqrt{x^2-6x+1}\right)^3 + \sum_{k\geq 4}\frac{1}{2k}\left(\frac{\sqrt{x^2-6x+1}}{x-3}\right)^k .
\end{aligned}$$

Since $\frac{3-x}{16}\sqrt{x^2-6x+1} + \frac{1}{2(x-3)}\sqrt{x^2-6x+1} = -\frac{1}{16(x-3)}\left(x^2-6x+1\right)^{\frac{3}{2}}$ it follows that

$$D(x) = \left(-\frac{1}{16(x-3)} + \frac{1}{6(x-3)^3}\right)\left(\sqrt{x^2-6x+1}\right)^3 + \sum_{k\geq 4}\frac{1}{2k}\left(\frac{\sqrt{x^2-6x+1}}{x-3}\right)^k + \tilde{A}(x),$$

where $\tilde{A}(x)$ is again analytic at 0 with radius of convergence $> \delta$. Writing $\sqrt{x^2-6x+1}$ as $\sqrt{1-\frac{x}{\delta}}\sqrt{1-\delta x}$ and by Theorem 1.29 we obtain

$$\begin{aligned}
d_n &= \left(-\frac{1}{16(\delta-3)} + \frac{1}{6(\delta-3)^3}\right)\left(\sqrt{1-\delta^2}\right)^3 \frac{1}{\Gamma\left(-\frac{3}{2}\right)} n^{-\frac{5}{2}}\delta^{-n}\left(1 + O\left(\frac{1}{n}\right)\right) \\
&\sim \frac{\left(3\sqrt{2}-4\right)^{\frac{3}{2}}}{8\sqrt{2\pi}} n^{-\frac{5}{2}}\left(3+2\sqrt{2}\right)^n \\
&\approx 0.00596026 n^{-\frac{5}{2}} 5.82843^n.
\end{aligned}$$

□

2.4 Rooted at a vertex

Vertex rooted dissections can be constructed from outer-edge rooted dissections by similar considerations as we have done for face rooted dissections, in fact, the cycle index sum for oriented vertex rooted and oriented outer-edge rooted dissections agree. But, since we already know the cycle index sum for dissections in general, we can apply Theorem 1.14 to obtain the cycle index sum for the rooted structures. However, the explicit construction of vertex rooted dissections gives a method to check the correctness of the derived cycle index sums.

Theorem 2.16 (vertex rooted dissections). *The cycle index sum for vertex rooted dissections is given by*

$$Z(\mathcal{V}) = \frac{s_1}{8s_2^2}\left(s_2 + s_1 - 3s_1s_2 + s_1s_2^2 - 2s_2^2 - (s_1+s_2)\sqrt{s_2^2 - 6s_2 + 1} - s_2^2\sqrt{s_1^2 - 6s_1 + 1}\right).$$

Proof. Theorem 1.14 yields

$$Z(\mathcal{V}) = s_1 \frac{\partial}{\partial s_1} Z(\mathcal{D})$$

and the rest is done using the command SIMPLIFY in Mathematica. □

Corollary 2.17. *The ordinary generating function of vertex rooted dissections is given by*

$$V(x) = \frac{1}{8x^2}\left(1 + x - 3x^2 - 2x^3 + x^4 - (1+x)\sqrt{x^4 - 6x^2 + 1} - x^3\sqrt{x^2 - 6x + 1}\right)$$

and its coefficients have the asymptotic estimate

$$v_n \sim \frac{(3\sqrt{2}-4)^{\frac{3}{2}}}{8\sqrt{2\pi}} n^{-\frac{3}{2}} \left(3 + 2\sqrt{2}\right)^n$$

Proof. $V(x)$ is obtained by Pólya's theorem. It can be split up as

$$V(x) = -\frac{x}{8}\sqrt{x^2 - 6x + 1} + A(x)$$

where $A(x)$ is analytic at 0 with radius of convergence $> \delta$. As before,

$$-\frac{x}{8}\sqrt{x^2 - 6x + 1} = -\frac{x}{8}\sqrt{1 - \delta x}\sqrt{1 - \frac{x}{\delta}},$$

and thus by Theorem 1.29

$$v_n \sim \frac{-\frac{\delta}{8}\sqrt{1-\delta^2}}{\Gamma\left(-\frac{1}{2}\right)} n^{-\frac{3}{2}} \delta^{-n}.$$ □

Remark. Bodirsky and Kang derived in [7, Sec. 5] recursive formulas for the number of reflective vertex rooted dissections on n vertices,

$$\begin{aligned} v_1^{\mathrm{r}} = v_2^{\mathrm{r}} = v_3^{\mathrm{r}} &= 1, \\ v_{2n}^{\mathrm{r}} = v_{2n+1}^{\mathrm{r}} &= v_{2n-1}^{\mathrm{r}} + 2\sum_{i=2}^{n} s_i v_{2n+1-2i}^{\mathrm{r}}, \quad n \geq 2, \end{aligned}$$

where s_i is the i-th bracketing number [31, A001003], and non-reflective (i.e., not invariant under reflection) vertex rooted dissections,

$$\begin{aligned} v_1^{\mathrm{nr}} &= 1, \\ v_n^{\mathrm{nr}} &= \frac{1}{2}\left(s_{n-1} - v_n^{\mathrm{r}}\right), \quad n \geq 2, \end{aligned}$$

Note that here also the single vertex is included and, for convenience, we will also include the empty graph into the counting, $v_0^{\mathrm{r}} := 1$, $v_0^{\mathrm{nr}} := 1$.

These recursion formulas can be used to verify the correctness of $V(x)$. We start to compute the ordinary generating function for reflective vertex rooted dissections. Let

$$V^{\mathrm{r}}(x) := \sum_{n\geq 0} v_n^{\mathrm{r}} x^n, \qquad S(x) := \sum_{n\geq 0} s_n x^n,$$

and $v_k^{\mathrm{r}} := 0$ for $k < 0$. Then we obtain (with $v_{2n-1}^{\mathrm{r}} = v_{2n-2}^{\mathrm{r}}$, ...)

$$\begin{aligned} v_{2n}^{\mathrm{r}} x^{2n} &= x^2 v_{2n-2}^{\mathrm{r}} x^{2n-2} + 2 \sum_{i=2}^{n} s_i x^{2i} v_{2n-2i}^{\mathrm{r}} x^{2n-2i}, \\ v_{2n+1}^{\mathrm{r}} x^{2n+1} &= x^2 v_{2n-1}^{\mathrm{r}} x^{2n-1} + 2 \sum_{i=2}^{n} s_i x^{2i} v_{2n-2i+1}^{\mathrm{r}} x^{2n-2i+1}, \end{aligned}$$

and thus

$$\sum_{n\geq 2} \left(v_{2n}^{\mathrm{r}} x^{2n} + v_{2n+1}^{\mathrm{r}} x^{2n+1} \right) = \\ x^2 \sum_{n\geq 2} \left(v_{2n-2}^{\mathrm{r}} x^{2n-2} + v_{2n-1}^{\mathrm{r}} x^{2n-1} \right) + 2 \sum_{i\geq 2} s_i x^{2i} \sum_{n\geq 2} \left(v_{2n-2i}^{\mathrm{r}} x^{2n-2i} + v_{2n-2i+1}^{\mathrm{r}} x^{2n-2i+1} \right),$$

that is

$$V^{\mathrm{r}}(x) - \sum_{k=0}^{3} v_k^{\mathrm{r}} x^k = x^2 \left(V^{\mathrm{r}}(x) - x - 1 \right) + 2 \left(S\left(x^2\right) - x^2 \right) V^{\mathrm{r}}(x).$$

Solving this equation for $V^{\mathrm{r}}(x)$ and using $S(x) = \frac{1}{4}\left(1 + x - \sqrt{x^2 - 6x + 1}\right)$ we obtain

$$V^{\mathrm{r}}(x) = \frac{(1+x)\left(1 + x^2 - \sqrt{x^4 - 6x^2 + 1}\right)}{4x^2}.$$

It is much easier to derive $V^{\mathrm{nr}}(x)$,

$$V^{\mathrm{nr}}(x) = \sum_{n\geq 0} v_n^{\mathrm{nr}} x^n = \frac{1}{2}\left(1 + x + xS(x) - V^{\mathrm{r}}(x)\right),$$

and thus we obtain

$$\begin{aligned} V^{\mathrm{nr}}(x) + V^{\mathrm{r}}(x) &= \frac{1+x}{2} + \frac{1}{2} x S(x) + \frac{1}{2} V^{\mathrm{r}}(x) \\ &= \frac{1+x}{2} + \frac{x}{8}\left(1 + x - \sqrt{x^2 - 6x + 1}\right) + \frac{1+x}{8x^2}\left(1 + x^2 - \sqrt{x^4 - 6x^2 + 1}\right) \end{aligned}$$

as generating function for vertex rooted dissections, which is the same as $1 + x + V(x)$.

2.5 Exact numbers

From the preceding ordinary generating functions we can compute the first coefficients by power series expansion to obtain exact numbers for several kinds of dissections. For some of them we derived also asymptotic estimates (Corollaries 2.2, 2.15, and 2.17), which we give (rounded to the closest integer) in the Tables 2.2 and 2.3 for comparison as well. The computations were done by Mathematica.

n	$(e_{\mathrm{o}}^{\mathrm{o}})_n$		$(e_{\mathrm{o}}^{\mathrm{r}})_n$	$(e_i)_n$	$(e_s)_n$
	exact	estimate			
2	1	0	1	0	0
3	1	0	1	0	0
4	3	2	1	1	1
5	11	7	3	2	0
6	45	32	3	11	3
7	197	147	11	42	0
8	903	702	11	211	11
9	4279	3427	45	1010	0
10	20793	17053	45	5153	45
11	103049	86153	197	26292	0
12	518859	440697	197	136602	197
13	2646723	2277972	903	713980	0
14	13648869	11880170	903	3763238	903
15	71039373	62435154	4279	19948914	0
16	372693519	330321979	4279	106348338	4279
17	1968801519	1757905135	20793	569656892	0
18	10463578353	9403974217	20793	3064826438	20793
19	55909013009	50540688821	103049	16554100920	0
20	300159426963	272758275041	103049	89735698719	103049
21	1618362158587	1477560788623	518859	488028170622	0
22	8759309660445	8031407260122	518859	2662113436741	518859
23	47574827600981	43791042753039	2646723	14561433290490	0
24	259215937709463	239448184673156	2646723	79851738590605	2646723
25	1416461675464871	1312712948502147	13648869	438918607278558	0

Table 2.2: Exact numbers for several kinds of edge rooted dissections. (Corollaries 2.2, 2.4, 2.6, and 2.8)

n	f_n	v_n		d_n	
		exact	estimate	exact	estimate
2	0	1	0	1	0
3	1	1	0	1	0
4	2	3	1	2	0
5	5	7	4	3	1
6	17	28	16	9	3
7	62	104	74	20	11
8	275	474	351	75	44
9	1272	2162	1713	262	190
10	6225	10495	8527	1117	853
11	31075	51623	43077	4783	3916
12	158376	259881	220349	21971	18362
13	816229	1323813	1138986	102249	87614
14	4251412	6826574	5940085	489077	424292
15	22319056	35521826	31217577	2370142	2081172
16	117998524	186357156	165160990	11654465	10322562
17	627573216	984411156	878952567	57916324	51703092
18	3355499036	5231840701	4701987108	290693391	261221506
19	18025442261	27954558029	25270344410	1471341341	1330018127
20	97239773408	150079972911	136379137520	7504177738	6818956876
21	526560862829	809181338723	738780394312	38532692207	35180018777
22	2861189112867	4379656153584	4015703630061	199076194985	182531983185
23	15595669996482	23787415123852	21895521376520	1034236705992	951979190283
24	85252072993968	129607975679166	119724092336578	5400337050086	4988503847357
25	467247847612316	708230844556870	656356474251073	28329240333758	26254258970043

Table 2.3: Exact numbers for face rooted f_n, vertex rooted v_n and unrooted dissections d_n and asymptotic estimates for vertex rooted and unrooted dissections. (Corollaries 2.12, 2.17, and 2.15)

Chapter 3

Outerplanar graphs

3.1 Connected outerplanar graphs

Connected outerplanar graphs have the property that their two-connected components (i.e., blocks) can be identified with dissections of convex polygons. The knowledge of the cycle index sum for such dissections, rooted and unrooted, allows us now the construction of the cycle index sums for rooted and unrooted connected outerplanar graphs (by using the Composition theorem 1.11). We want to emphasize again that ordinary generating functions are insufficient for the composition of unlabelled structures since they do not inherit essential information about possible automorphisms.

Using the theory in the first chapter about rooted connected graphs with given blocks (Section 1.2.4) and about the unrooting of such graphs (Section 1.2.5) it remains to insert the cycle index sums for vertex rooted and unrooted dissections in the formulas from the first chapter. Thereby we obtain an implicit formula for vertex rooted connected outerplanar graphs, which is also suitable for the computation of exact numbers. Furthermore, we can analyze this equation to determine the point where the ordinary generating function for vertex rooted connected outerplanar graphs ceases to be analytic, and thus we can estimate asymptotics. The exact determination of the growth-constant for vertex rooted and unrooted connected outerplanar graphs seems to be impossible. However, we can approximate it.

Notation. The set of unrooted unlabelled connected outerplanar graphs is denoted by $\mathcal{C}$, the set of vertex rooted unlabelled connected outerplanar graphs by $\hat{\mathcal{C}}$. Since we only analyze graphs which are unrooted or rooted on a vertex, we will just say rooted graphs for vertex rooted graphs from now on. Ordinary generating functions are denoted by capital letters again and counts by small letters, $\hat{C}(x) = \sum_n \hat{c}_n x^n$ and $C(x) = \sum_n c_n x^n$.

3.1.1 Counting formulas and exact numbers

Theorem 3.1 (rooted connected outerplanar graphs)**.** *The cycle index sum for vertex rooted connected outerplanar graphs is implicitly determined by the equation*

$$Z\left(\hat{\mathcal{C}}\right) = s_1 \exp\left(\sum_{k\geq 1} \frac{Z\left(\mathcal{V}; Z\left(\hat{\mathcal{C}}; s_k, s_{2k}\right), Z\left(\hat{\mathcal{C}}; s_{2k}, s_{4k}\right), \ldots\right)}{k\, Z\left(\hat{\mathcal{C}}; s_k, s_{2k}\right)}\right) \tag{3.1}$$

and the ordinary generating function satisfies

$$\hat{C}(x) = x \exp\left(\sum_{k\geq 1} \frac{Z\left(\mathcal{V}; \hat{C}\left(x^k\right)\right)}{k\, \hat{C}\left(x^k\right)}\right). \tag{3.2}$$

Proof. Formula (3.1) is an immediate consequence of Formula (1.7), page 17, for the cycle index sum of rooted connected graphs with given blocks. Formula (3.2) follows from Pólya's theorem by application of the mapping $s_k \mapsto x^k$. □

Theorem 3.2 (connected outerplanar graphs). *The cycle index sum for connected outerplanar graphs is given by*

$$Z(\mathcal{C}) = Z\left(\hat{\mathcal{C}}\right) + Z\left(\mathcal{D}; Z\left(\hat{\mathcal{C}}\right)\right) - Z\left(\mathcal{V}; Z\left(\hat{\mathcal{C}}\right)\right) \tag{3.3}$$

and the ordinary generating function by

$$C(x) = \hat{C}(x) + Z\left(\mathcal{D}; \hat{C}(x)\right) - Z\left(\mathcal{V}; \hat{C}(x)\right). \tag{3.4}$$

Proof. Formula (3.3) is an immediate consequence of Theorem 1.16 about unrooting of graphs with given blocks. Formula (3.4) is by Pólya again. □

3.1.1.1 Exact numbers

Recall that in the right hand side of (3.2) the recipe for the construction of a rooted connected outerplanar graph is encoded. Having the knowledge about the number of connected outerplanar graphs of up to (say) m vertices, this formula "constructs" connected outerplanar graphs with up to $m+1$ vertices. It can therefore be used to compute the coefficients of $\hat{C}(x)$ one-by-one: Denote by $\hat{C}^{[m]}(x)$ the ordinary generating function $\hat{C}(x)$ chopped after the term of order m,

$$\hat{C}^{[m]}(x) := \sum_{n=0}^{m} \hat{c}_n x^n,$$

and assume $\hat{C}^{[m]}(x)$ to be explicitely known. Then $\hat{c}_{m+1}$ can be determined by replacing $\hat{C}(x)$ by $\hat{C}^{[m]}(x)$ in the right hand side of (3.2) and computation of the coefficient of x^{m+1},

$$\hat{c}_{m+1} = \left[x^{m+1}\right] x \exp\left(\sum_{k=1}^{m} \frac{Z\left(\mathcal{V}; \hat{C}^{[m]}\left(x^k\right)\right)}{k\, \hat{C}^{[m]}\left(x^k\right)}\right).$$

The computation for the unrooted case becomes then easier again due to the simpleness of Formula (3.4). In Table 3.1 we have also included the number of rooted and unrooted dissections for comparison. The computations were done by Mathematica.

3.1.2 Asymptotics

As pointed out in the first chapter, asymptotic estimates for the coefficients of an analytic function can be obtained by examination of the dominant singularities of the function. The counting functions in the second chapter were explicitely given and thus their dominant singularities could be easily found. For rooted connected outerplanar graphs the function $\hat{C}(x)$ is determined by the equation (3.2) and can (in theory) be computed by inversion of this equation. It ceases to be analytic at a point where this inversion is not possible due to singularities in other components of equation (3.2) or where the inversion is not unique. We conjecture here that the dominant singularity of $\hat{C}(x)$ is caused by a missing uniqueness of the inversion, i.e., the implicit function theorem, which ensures the existence of a unique solution of equation (3.2), cannot be applied. This conjecture will be verified numerically in the next section.

We can make the following estimates for the growth-ratio of the number of rooted connected outerplanar graphs: The asymptotic number of labelled connected outerplanar graphs has been found by Bodirsky et al. in [3]. When we let l_n to be the number of labelled connected outerplanar

n	v_n	d_n	$\hat{c}_n$	c_n
1	0	0	1	1
2	1	1	1	1
3	1	1	3	2
4	3	2	10	5
5	7	3	40	13
6	28	9	181	46
7	104	20	918	172
8	474	75	5039	777
9	2162	262	29313	3783
10	10495	1117	177773	20074
11	51623	4783	1110517	111604
12	259881	21971	7093110	646409
13	1323813	102249	46079944	3846640
14	6826574	489077	303422249	23410035
15	35521826	2370142	2020011842	144965988
16	186357156	11654465	13572042838	910898943
17	984411156	57916324	91901417553	5794179218
18	5231840701	290693391	626498987843	37248630398
19	27954558029	1471341341	4296043185620	241676806702
20	150079972911	7504177738	29611845187329	1580880366039
21	809181338723	38532692207	205050800043677	10416314047854
22	4379656153584	199076194985	1425764035634376	69080674190341
23	23787415123852	1034236705992	9950544932379738	460841447382976
24	129607975679166	5400337050086	69679887165423882	3090747326749823
25	708230844556870	28329240333758	489438852540945514	20829976038652612

Table 3.1: Exact numbers for rooted dissections, unrooted dissections, rooted connected outerplanar graphs, and unrooted connected outerplanar graphs. (Corollaries 2.17 and 2.15, Theorems 3.1 and 3.2)

graphs, then they have shown that $l_n \sim \lambda^{-n+O(\log_\lambda n)} n!$ for $\lambda \approx 0.136593$. Therefore, the radius of convergence of $L(x) := \sum_n \frac{1}{n!} l_n x^n$ is λ. Since $n\, l_n \leq n!\, \hat{c}_n$ it follows that $xL'(x) \leq \hat{C}(x)$. Hence the radius of convergence of $\hat{C}(x)$ is bounded from above by the one of $xL'(x)$, which is λ. More precisely it can even be shown that this inequality is strict. Observe that when two components of an outerplanar graph are joined by an edge, the resulting graph is still outerplanar. McDiarmid et al. [25] have called such classes of graphs *addable* and it is easy to see that their result on the difference between the growth constants of unlabelled and labelled planar graphs [25, Cor. 4.7] can be adapted to the case of outerplanar graphs as well. (Actually we have this result for outerplanar graphs which are not necessarily connected, but, as we will see in the next section, the growth constants coincide with the one of the connected graphs.) Furthermore, to obtain a lower bound on the radius of convergence of $\hat{C}(x)$, recall that the number of unlabelled outerplanar graphs is dominated by the number of unlabelled planar graphs p_n. Hence, $\hat{c}_n \leq n p_n$. Bonichon et al. have shown in [9] that $p_n \leq 2^{\alpha n + O(\log_2 n)}$ for $\alpha \approx 4.9098$, which yields $\hat{c}_n \leq 2^{\alpha n + O(\log_2 n)}$. To sum up, we know the following estimate for the growth-constant of unlabelled rooted connected outerplanar graphs:

$$0.0333 \leq \frac{1}{\text{growth ratio of } \hat{c}_n} < 0.136593.$$

We now continue to determine the growth-ratio. As described above we hardly believe that the dominant singularity of $\hat{C}(x)$ is caused by a failure of the implicit function theorem, and thus the value of $\hat{C}(x)$ is not uniquely determined by equation (3.2) for x with $|x|$ greater than the growth-ratio. This situation is described by Theorem 1.30 (page 25). Therefore, let

$$H(x,y) := x \exp\left(\frac{Z\left(\mathcal{V}; y, \hat{C}(x^2)\right)}{y} + \sum_{k\geq 2} \frac{Z\left(\mathcal{V}; \hat{C}(x^k), \hat{C}(x^{2k})\right)}{k\,\hat{C}(x^k)}\right) - y \tag{3.5}$$

and observe that due to equation (3.2),

$$H\left(x, \hat{C}(x)\right) = 0.$$

To apply Theorem 1.30, $H(x,y)$ has to allow for a power series expansion around $(0,0)$ which convergences for $|x| < R$ and $|y| < S$ with radiuses of convergence $R, S > 0$. These numbers are determined by the dominant singularities of $Z(\mathcal{V})$, that is, $H(x,y)$ converges for $|y| < \delta$ and x such that $\left|\hat{C}(x^l)\right| < \delta$, $l \geq 2$. Since $\hat{C}(x)$ is strictly increasing for positive x, $\left|\hat{C}(x^l)\right| \leq \left|\hat{C}(x^2)\right|$ for all $l \geq 2$. Therefore, the expansion of $H(x,y)$ around $(0,0)$ converges for $|x| < R := \sqrt{\hat{C}^{-1}(\delta)}$ and $|y| < S := \delta$.

Conjecture 3.3. *The system (1.15) permits a solution (ρ, σ) with $\rho \in (0, R)$ and $\sigma \in (0, S)$, that is*

$$H(\rho,\sigma) = 0 \quad \textit{and} \quad \frac{\partial}{\partial y} H(\rho,\sigma) = 0.$$

Furthermore, $\frac{\partial^2}{\partial y^2} H(\rho,\sigma) \neq 0$ and $\frac{\partial}{\partial x} H(\rho,\sigma) \neq 0$.

Theorem 3.4 (asymptotic number of rooted connected outerplanar graphs)**.** *If Conjecture 3.3 is true for $H(x,y)$ from (3.5), then the function $\hat{C}(x)$ has an asymptotic expansion of the form*

$$\hat{C}(x) = \sum_{k\geq 0} \hat{C}_k X^k, \qquad X := \sqrt{1 - \frac{x}{\rho}}, \tag{3.6}$$

with

$$\hat{C}_0 = \sigma, \quad \hat{C}_1 = -\sqrt{\frac{2\rho \frac{\partial}{\partial x} H(\rho,\sigma)}{\frac{\partial^2}{\partial y^2} H(\rho,\sigma)}},$$

and constants $\hat{C}_k$, $k \geq 2$, which can be computed from the derivatives of $H(x,y)$ at (ρ,σ). Furthermore, $\hat{c}_n$ has the asymptotic expansion

$$\hat{c}_n = \sum_{k\geq 0} \binom{n + k - \frac{1}{2}}{n} \hat{C}_{2k+1} \rho^{-n}$$

and particularly

$$\hat{c}_n \sim -\frac{\hat{C}_1}{2\sqrt{\pi}} n^{-\frac{3}{2}} \rho^{-n}.$$

Proof. As checked before, $H(x,y)$ is a bivariate analytic function around $(0,0)$. It satisfies $H(0,0) = 0$ and $\frac{\partial}{\partial y} H(0,0) = -1$ and from Conjecture 3.3 it follows that $H_{0,0} = 0$, $H_{0,1} = 0$, $H_{0,2} \neq 0$, and $H_{1,0} \neq 0$, where $H_{i,j} := \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} H(x,y)\big|_{(x,y)=(r,s)}$ again. The function $\hat{C}(x)$ fulfills $H\left(x, \hat{C}(x)\right) = 0$, is analytic at 0 (since the growth-ratio of $\hat{c}_n$ is bounded from below by 0.0333), has nonnegative coefficients, and is aperiodic (since $\hat{C}_n > 0$ for all $n \geq 1$).

So we can apply Theorem 1.30 on positive implicit functions which yields the claimed expansions and asymptotic estimates. □

We can now proceed with the asymptotic expansion of the ordinary generating function for unrooted connected outerplanar graphs.

Theorem 3.5 (asymptotic number of unrooted connected outerplanar graphs)**.** *If Conjecture 3.3 is true, then the function $C(x)$ has an asymptotic expansion of the form*

$$C(x) = C(\rho) + \sum_{k\geq 2} C_k X^k, \qquad X := \sqrt{1-\frac{x}{\rho}}, \tag{3.7}$$

with constants C_k, $k \geq 2$, which can be computed from the constants $\hat{C}_k$, and ρ is the same as in Theorem 3.4. Furthermore, c_n has the asymptotic expansion

$$c_n = \sum_{k\geq 1} \binom{n+k-\frac{1}{2}}{n} C_{2k+1}\rho^{-n}$$

and particularly

$$c_n \sim \frac{3C_3}{4\sqrt{\pi}} n^{-\frac{5}{2}}\rho^{-n}.$$

Proof. Since $\sigma < \delta$ it is clear that the dominant singularity of $C(x)$ is determined by $\hat{C}(x)$ (compare Formula (3.4)). Hence, the asymptotic expansion of $C(x)$ around ρ can be obtained by composing the analytic expansion of $C(x)$ around $\sigma = \hat{C}(\rho)$ with the asymptotic expansion of $\hat{C}(x)$. That is, in (3.4) we substitute $\hat{C}(x)$ by (3.6) and x^k by $\rho^k\left(1-X^2\right)^k$, $k \geq 2$, and thus

$$\begin{aligned} C(x) \quad = \quad & \sum_{k\geq 0} \hat{C}_k X^k + Z\left(\mathcal{D}; \sum_{k\geq 0} \hat{C}_k X^k, \hat{C}\left(\rho^2\left(1-X^2\right)^2\right), \hat{C}\left(\rho^3\left(1-X^2\right)^3\right), \ldots\right) \\ & -Z\left(\mathcal{V}; \sum_{k\geq 0} \hat{C}_k X^k, \hat{C}\left(\rho^2\left(1-X^2\right)^2\right), \hat{C}\left(\rho^3\left(1-X^2\right)^3\right), \ldots\right) \end{aligned} \tag{3.8}$$

and develop in terms of X (around $X=0$) which yields the asymptotic expansion

$$C(x) = \sum_{k\geq 0} C_k X^k. \tag{3.9}$$

It remains to verify that $C_1 = 0$ and $C_3 \neq 0$. From (3.8) it is clear that

$$C_1 = \hat{C}_1 + \frac{\partial}{\partial s_1} Z(\mathcal{D})\,\hat{C}_1 - \frac{\partial}{\partial s_1} Z(\mathcal{V})\,\hat{C}_1\Big|_{(s_1,s_2)=\left(\sigma,\hat{C}(\rho^2)\right)}.$$

From Theorem 1.14 (page 18) we know

$$s_1 \frac{\partial}{\partial s_1} Z(\mathcal{D}) = Z(\mathcal{V}),$$

and thus

$$C_1 = \hat{C}_1\left(1 + \frac{Z\left(\mathcal{V}; \sigma, \hat{C}\left(\rho^2\right)\right)}{\sigma} - \frac{\partial}{\partial s_1} Z(\mathcal{V})\Big|_{(s_1,s_2)=\left(\sigma,\hat{C}(\rho^2)\right)}\right).$$

Now recall the definition of $H(x,y)$ in (3.5) and Conjecture 3.3. From there we obtain

$$\begin{aligned}
0 &= \frac{\partial}{\partial y} H(\rho,\sigma) \\
&= (H(\rho,\sigma)+\sigma)\,\frac{\partial}{\partial y}\left.\frac{Z\left(\mathcal{V};y,\hat{C}\left(x^2\right)\right)}{y}\right|_{(x,y)=(\rho,\sigma)} - 1 \\
&= \sigma\,\frac{1}{\sigma^2}\left(\frac{\partial}{\partial s_1} Z\left(\mathcal{V};y,\hat{C}\left(\rho^2\right)\right)\Big|_{y=\sigma}\sigma - Z\left(\mathcal{V};\sigma,\hat{C}\left(\rho^2\right)\right)\right) - 1 \\
&= \frac{\partial}{\partial s_1} Z(\mathcal{V})\Big|_{(s_1,s_2)=\left(\sigma,\hat{C}\left(\rho^2\right)\right)} - \frac{1}{\sigma} Z\left(\mathcal{V};\sigma,\hat{C}\left(\rho^2\right)\right) - 1 \\
&= -\frac{1}{\hat{C}_1} C_1,
\end{aligned}$$

which proves $C_1 = 0$.

The expression for C_3 is more complex and due to the absence of analytic expressions for ρ and σ we are not able to give an analytic expression for C_3. However, we can use the following argument: From the asymptotic expansion (3.9) of $C(x)$ and Theorem 1.29 it follows that

$$c_n = \frac{C_{2k+1}}{\Gamma\left(-k-\frac{1}{2}\right)} n^{-k-\frac{3}{2}} \rho^{-n}\left(1 + O\left(\frac{1}{n}\right)\right)$$

with $C_{2k+1} \neq 0$ for some $k \geq 1$. Furthermore, it is clear that $nc_n \geq \hat{c}_n$. Since $\hat{c}_n \sim -\frac{\hat{C}_1}{2\sqrt{\pi}} n^{-\frac{3}{2}} \rho^{-n}$ (Theorem 3.4), we obtain $k = 1$, i.e. $C_3 \neq 0$. This proves the claimed expansion. □

3.1.3 Approximation of the growth-constant

Unfortunately, we were not able to solve system (1.15) analytically. As far as we know, this is also the case for simpler structures, e.g., trees [16, Sec. VII.2.3]. The only simplification we were able to find is to reduce the system (1.15) by one variable. Recall from the proof of Theorem 3.5 that

$$\frac{\partial}{\partial y} H(\rho,\sigma) = \frac{\partial}{\partial s_1} Z(\mathcal{V})\Big|_{(s_1,s_2)=\left(\sigma,\hat{C}\left(\rho^2\right)\right)} - \frac{1}{\sigma} Z\left(\mathcal{V};\sigma,\hat{C}\left(\rho^2\right)\right) - 1.$$

Thus using the explicit formula for $Z(\mathcal{V})$ (given by Theorem 2.16 on page 42) the equation $\frac{\partial}{\partial y} H(\rho,\sigma) = 0$ simplifies to

$$\sigma\left(1 + \hat{C}\left(\rho^2\right)\left(\hat{C}\left(\rho^2\right) - 3\right) - \frac{\hat{C}\left(\rho^2\right)^2(\sigma-3)}{\sqrt{\sigma^2-6\sigma+1}} - \sqrt{\hat{C}\left(\rho^2\right)^2 - 6\hat{C}\left(\rho^2\right) + 1}\right) = 8\hat{C}\left(\rho^2\right)^2.$$

When we let

$$q_1 = \frac{1}{\sqrt{\sigma^2-6\sigma+1}} \quad \text{and} \quad q_2 = \sqrt{\hat{C}\left(\rho^2\right)^2 - 6\hat{C}\left(\rho^2\right) + 1},$$

we obtain the following system of equations,

$$\begin{aligned}
\sigma\left(1 + \hat{C}\left(\rho^2\right)\left(\hat{C}\left(\rho^2\right) - 3\right) - \hat{C}\left(\rho^2\right)^2(\sigma-3)\,q_1 - q_2\right) - 8\hat{C}\left(\rho^2\right)^2 &= 0, && (3.10a) \\
q_1^2\left(\sigma^2-6\sigma+1\right) - 1 &= 0, && (3.10b) \\
q_2^2 - \left(\hat{C}\left(\rho^2\right)^2 - 6\hat{C}\left(\rho^2\right) + 1\right) &= 0, && (3.10c)
\end{aligned}$$

which is polynomial in the variables q_1, q_2, and σ. By algebraic elimination, we can reduce this system to one equation in the variable σ. This is done by an approach based on resultants.

Definition 3.6 (Sylvester matrix, resultant)**.** Given two polynomials

$$P(x) = \sum_{j=0}^{m} a_j x^j \quad \text{and} \quad Q(x) = \sum_{k=0}^{n} b_k x^k$$

of degree m and n, respectively, with coefficients from some field $\mathbb{K}$, the *Sylvester matrix* is the following $(m+n) \times (m+n)$-matrix

$$\begin{bmatrix} a_m & a_{m-1} & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & a_m & a_{m-1} & \cdots & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & a_m & a_{m-1} & \cdots & a_0 \\ b_n & b_{n-1} & \cdots & b_0 & 0 & \cdots & 0 \\ 0 & b_n & b_{n-1} & \cdots & b_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & b_n & b_{n-1} & \cdots & b_0 \end{bmatrix}$$

and the *resultant* (with respect to variable x) of $P(x)$ and $Q(x)$, denoted by $R(P(x), Q(x), x)$, is the determinant of the Sylvester matrix.

The main property of resultants is the following [16, App. B.1]: If $P(x)$ and $Q(x)$ have a common root in the algebraic closure $\bar{\mathbb{K}}$ of $\mathbb{K}$, then $R(P(x), Q(x), x) = 0$. Conversely, if $R(P(x), Q(x), x) = 0$ holds, then either $a_m = 0$ or $b_n = 0$ or $P(x)$ and $Q(x)$ have a common root in $\bar{\mathbb{K}}$. (The idea is that the vector $\left(\xi^{m+n-1}, \ldots, \xi^2, \xi, 1\right)$ belongs to the kernel of the Sylvester matrix, if ξ is a common root of $P(x)$ and $Q(x)$.)

We can now use the resultant to eliminate the variables q_1 and q_2 in the polynomial system (3.10). Computing the resultant of the polynomials on the left hand side of (3.10a) and (3.10b) with respect to q_1 gives a polynomial in the variables q_2 and σ, whose roots are the solution of the system (3.10a), (3.10b). Application of the same method to the just computed resultant and the polynomial on the left hand side of (3.10c) with respect to q_2 gives a polynomial in σ, whose roots are the solutions of the system (3.10). $R(R((3.10a), (3.10b), q_1), (3.10c), q_2)$ equals

$$\begin{aligned} & 4096t^8 \\ & + \left(-2048t^6 + 6144t^7 - 51200t^8\right)\sigma \\ & + \left(256t^4 - 1536t^5 + 28672t^6 - 76032t^7 + 179456t^8 + 768t^9 - 128t^{10}\right)\sigma^2 \\ & + \left(-3392t^4 + 19584t^5 - 127616t^6 + 260352t^7 - 123808t^8 - 4896t^9 + 800t^{10}\right)\sigma^3 \\ & + \left(13632t^4 - 72072t^5 + 189324t^6 - 158136t^7 + 37088t^8 + 2412t^9 - 268t^{10} - 12t^{11} + t^{12}\right)\sigma^4 \\ & + \left(-16216t^4 + 62208t^5 - 84664t^6 + 42168t^7 - 6304t^8 - 432t^9 + 44t^{10}\right)\sigma^5 \\ & + \left(7860t^4 - 19680t^5 + 16996t^6 - 5940t^7 + 682t^8 + 24t^9 - 2t^{10}\right)\sigma^6 \\ & \left(-1520t^4 + 2592t^5 - 1600t^6 + 432t^7 - 44t^8\right)\sigma^7 \\ & \left(100t^4 - 120t^5 + 56t^6 - 12t^7 + t^8\right)\sigma^8, \end{aligned}$$

where $t = \hat{C}\left(\rho^2\right)$. Denote the roots of this polynomial by $\sigma_1(\rho), \ldots, \sigma_8(\rho)$. It remains now to solve the single equations

$$H(\rho, \sigma_i(\rho)) = 0, \qquad i = 1, \ldots, 8,$$

and to pick the correct solution ρ. Here, new problems arise. The function $H(x, y)$ depends on $\hat{C}(x)$, which we do not know explicitly, and includes also an infinite sum we were not able to simplify. We can therefore only approximate the solutions of $H(\rho, \sigma_i(\rho)) = 0$ by truncation of the infinite series in $H(x,y)$ at some index m and replacement of $\hat{C}(x)$ by some $\hat{C}^{[m]}(x) := \sum_{n=1}^{m} \hat{c}_n x^n$, such that the coefficients $\hat{c}_1, \ldots, \hat{c}_m$ are known. That is, we search for roots of the functions

$$H_i^{[m]}(\rho) := \rho \exp\left(\frac{Z\left(\mathcal{V}; \sigma_i(\rho), \hat{C}^{[m]}(\rho^2)\right)}{\sigma_i(\rho)} + \sum_{k=2}^{m} \frac{Z\left(\mathcal{V}; \hat{C}^{[m]}(\rho^k), \hat{C}^{[m]}(\rho^{2k})\right)}{k\, \hat{C}^{[m]}(\rho^k)} \right) - \sigma_i(\rho),$$

$i = 1, \ldots, 8$, in the interval $(0.0333, 0.136593)$.

It turns out that two of the $\sigma_i(\rho)$, $i = 1, \ldots, 8$, are not real numbers for $\rho \in (0.0333, 0.136593)$. For two other, $H_i^{[m]}(\rho)$ has no root in $(0.0333, 0.136593)$. The remaining four have roots in $(0.0333, 0.136593)$, but two of them do not satisfy the equation $\frac{\partial}{\partial y} H(\rho, \sigma) = 0$ (where again $\hat{C}(\rho^2)$ was substituted by $\hat{C}^{[m]}(\rho^2)$) and for one of the others, the obtained σ exceeds $S = \delta$. The remaining branch yields the desired root that satisfies the requirements of Conjecture 3.3, see Figure 3.1 for a plot.

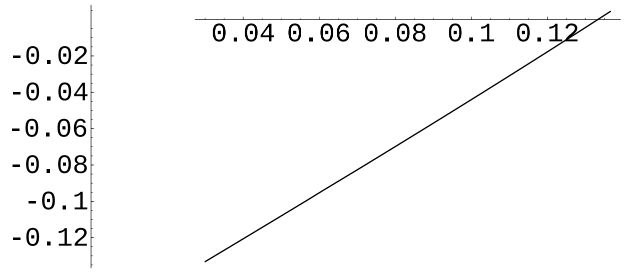

Figure 3.1: Plot of $H_3^{[4]}(\rho)$, which is the correct branch that satisfies the requirements of Conjecture 3.3.

Depending on m we obtain the following estimates of ρ and σ (by numerical solution of the equation $H_i^{[m]}(\rho) = 0$):

m	approximation of ρ	approximation of σ
1	**0.13**461876886110181369...	**0.1707560**0453372418417...
4	**0.1332**7064317786556821...	**0.170756019**47848944320...
8	**0.133269432**88029243729...	**0.17075601961**677905360...
16	**0.13326943266744682**071...	**0.17075601961680752650**...
25	**0.13326943266744680944**...	**0.17075601961680752650**...

For $m = 25$, the residuals in the equations $H_i^{[m]}(\rho, \sigma) = 0$ and $\frac{\partial}{\partial y} H_i^{[m]}(\rho, \sigma) = 0$ have an order of 10^{-58}. It is $R \approx \sqrt{\left(\hat{C}^{[m]}\right)^{-1}(\delta)} \approx 0.367608 > \rho$, $\sigma < \delta \approx 0.171573$ and $\frac{\partial^2}{\partial y^2} H(\rho, \sigma) \approx 549.359$, $\frac{\partial}{\partial x} H(\rho, \sigma) \approx 1.34975$. These approximated numbers support Conjecture 3.3. For the upcoming estimates of growth constants and asymptotic expansions, we use the values for $m = 25$ as approximations of ρ and σ.

The asymptotic expansion of $\hat{C}(x)$ takes now the form

$$\begin{aligned}\hat{C}(x) \approx\ & 0.170756 - 0.0255905X - 0.198284X^2 + 0.0362965X^3 \\ & + 0.0409361X^4 - 0.0174327X^5 - 0.0201252X^6 + 0.0111939X^7,\end{aligned}$$

where the coefficients $\hat{C}_2$ to $\hat{C}_7$ were computed using the formulas for Y_i in the proof of Theorem 1.30. Hence,

$$\hat{c}_n \sim -\frac{\hat{C}_1}{2\sqrt{\pi}} n^{-\frac{3}{2}} \rho^{-n} \approx 0.00721895\, n^{-\frac{3}{2}} 7.50360^n.$$

When we include also higher order terms, we can obtain more accurate values for the asymptotic estimates of $\hat{c}_n$, as shown in Table 3.3.

For connected outerplanar graphs, we can approximate the constants C_k the same way, where we chop the infinite sum introduced by the unrooted dissections (in $Z(\mathcal{D})$) at $d = 25$. Then,

$$\begin{aligned} C(x) \approx\ & 0.158617 - 0.201111X^2 + 0.0179720X^3 \\ & + 0.0390048X^4 - 0.00589968X^5 - 0.0153128X^6 + 0.00584822X^7, \end{aligned}$$

and thus

$$c_n \sim \frac{3C_3}{4\sqrt{\pi}} n^{-\frac{5}{2}} \rho^{-n} \approx 0.00760471\, n^{-\frac{5}{2}} 7.50360^n.$$

When we include also higher order terms we can obtain more accurate values for the asymptotic estimates of c_n, as shown in Table 3.2.

n	$s = 1$	$s = 2$	$s = 3$	c_n exact
1	0	0	0	1
2	0	0	1	1
3	0	1	0	2
4	1	2	7	5
5	5	7	10	13
6	22	27	34	46
7	106	125	144	172
8	546	627	690	777
9	2957	3331	3566	3783
10	16642	18463	19435	20074
11	96495	105811	110121	111604
12	573213	622731	642943	646409
13	3474012	3745540	3844704	3846640
14	21412658	22940733	23445648	23410035
15	133893281	142683925	145336002	144965988
16	847699683	899232059	913535269	910898943
17	5425384881	5732452815	5811368631	5794179218
18	35055744665	36911836304	37355950614	37248630398
19	228433077815	239794873174	242338062071	241676806702
20	1499810865010	1570145682102	1584935055452	1580880366039
21	9914216081387	10354019466177	10441212918461	10416314047854
22	65938606038199	68713695997539	69234133629522	69080674190341
23	440996596707159	458651010131809	461792112104866	460841447382976
24	2964366571610156	3077519386663540	3096669654673439	3090747326749823
25	20019068596971865	20749254199162695	20867084818336278	20829976038652612

Table 3.2: Asymptotic estimates $\sum_{k=1}^{s} \binom{n+k-\frac{1}{2}}{n} C_{2k+1}\rho^{-n}$ for the numbers of connected outerplanar graphs (rounded to the closest nonnegative integer) and the exact numbers c_n for comparison (Theorems 3.2 and 3.5).

n	$s=0$	$s=1$	$s=2$	$s=3$	$\hat{c}_n$ exact
1	0	0	0	0	1
2	0	1	0	2	1
3	1	2	4	0	3
4	3	6	8	18	10
5	17	27	32	39	40
6	94	138	153	167	181
7	552	766	823	860	918
8	3367	4469	4710	4830	5039
9	21054	27026	28129	28580	29313
10	134282	167893	173274	175134	177773
11	870200	1065082	1092611	1100860	1110517
12	5713423	6871090	7017409	7056096	7093110
13	37924537	44940695	45743019	45932825	46079944
14	254080722	297325995	301841230	302807670	303422249
15	1715867212	1986279763	2012254816	2017331058	2020011842
16	11668126964	13380151883	13532422440	13559799677	13572042838
17	79827655425	90784830822	91692171192	91843220994	91901417553
18	549078279552	619877301445	625361777975	626211840509	626498987843
19	3794793631229	4256139942137	4289712368555	4294580190987	4296043185620
20	26339003915820	29368040762190	29575869753730	29604177538262	29611845187329
21	183520301720586	203543176947053	204842730961932	205009624676857	205050800043677
22	1283171620477469	1416342056795547	1424542037566579	1425538187707592	1425764035634376
23	9000462377500233	9891104638943528	9943270821923919	9949283087334995	9950544932379738
24	63314845047491291	69301716979836337	69636066828389176	69672721637738360	69679887165423882
25	446583685046177413	487014447234202332	489172037793418976	489397572959771289	489438852540945514

Table 3.3: Asymptotic estimates $\sum_{k=0}^{s} \binom{n+k-\frac{1}{2}}{n} \hat{C}_{2k+1}\rho^{-n}$ for the numbers of rooted connected outerplanar graphs (rounded to the closest nonnegative integer) and the exact numbers $\hat{c}_n$ for comparison (Theorems 3.1 and 3.4).

3.2 Outerplanar graphs

An outerplanar graph is a collection of connected outerplanar graphs. This simple property makes it easy to formulate the cycle index sum for outerplanar graphs. We will denote the set of unlabelled outerplanar graphs by $\mathcal{G}$, its ordinary generating function by $G(x)$ and the number of outerplanar graphs on n vertices by g_n. Using the exact numbers and the asymptotic expansion of the ordinary generating function of connected outerplanar graphs we can also compute exact numbers for outerplanar graphs and an asymptotic expansion of their generating function. Furthermore, we can relate the asymptotic estimates for the numbers of connected outerplanar graphs and not necessarily connected outerplanar graphs to obtain (asymptotically) the probability to be connected, the expected number of components and the expected number of components of a specific type, especially the expected number of isolated vertices.

3.2.1 Counting formula, exact numbers and asymptotics

Theorem 3.7 (outerplanar graphs). *The cycle index sum for outerplanar graphs is given by*

$$Z(\mathcal{G}) = \left(\exp \sum_{k \geq 1} \frac{Z(\mathcal{C}; s_k, s_{2k}, \ldots)}{k}\right) - 1$$

and the ordinary generating function by

$$G(x) = \left(\exp \sum_{k \geq 1} \frac{1}{k} C\left(x^k\right)\right) - 1. \tag{3.11}$$

Proof. The permutation group of a collection of k objects is the symmetric group S_k, its cycle index sum is $Z(S_k)$. Since an outerplanar graph is a collection of at least one connected outerplanar graphs we obtain (by use of Formula (1.1))

$$Z(\mathcal{G}) = \sum_{k \geq 1} Z(S_k)\left[Z(\mathcal{C})\right] = \left(\exp \sum_{k \geq 1} \frac{Z(\mathcal{C}; s_k, s_{2k}, \ldots)}{k}\right) - Z(S_0)\left[Z(\mathcal{C})\right].$$

The ordinary generating function is given by Pólya's theorem. □

Exact numbers of outerplanar graphs can be computed from (3.11) by power series expansion using the exact numbers for connected outerplanar graphs. The exact numbers for outerplanar graphs with up to 25 vertices is given in Table 3.4.

Theorem 3.8 (asymptotic number of outerplanar graphs). *If Conjecture 3.3 is true, then the function $G(x)$ has an asymptotic expansion of the form*

$$G(x) = G(\rho) + \sum_{k \geq 2} G_k X^k, \qquad X := \sqrt{1 - \frac{x}{\rho}},$$

with $G_3 = (1 + G(\rho))\, C_3$ and constants G_k, $k \geq 2$, which can be computed from the constants C_k, and ρ as in Theorem 3.4. Furthermore, g_n has the asymptotic expansion

$$g_n = \sum_{k \geq 1} \binom{n + k - \frac{1}{2}}{n} G_{2k+1} \rho^{-n},$$

and particularly

$$g_n \sim \frac{3G_3}{4\sqrt{\pi}} n^{-\frac{5}{2}} \rho^{-n}.$$

Proof. The proof is similar to the one of the asymptotic expansion of $C(x)$ (Theorem 3.5). The dominant singularity of $G(x)$ is determined by $C(x)$, so we replace $C(x)$ by its asymptotic expansion (3.7) and x^k by $\rho^k\left(1-X^2\right)^k$ for $k \geq 2$,

$$\begin{aligned}
G(x) &= \exp\left(C(\rho)+\sum_{k\geq 2} C_k X^k+\sum_{k\geq 2}\frac{1}{k}C\left(\rho^k\left(1-X^2\right)^k\right)\right)-1 \\
&= \exp\left(C(\rho)+C_2X^2+C_3X^3+O\left(X^4\right)+\sum_{k\geq 2}\frac{1}{k}C\left(\rho^k\right)+aX^2+O\left(X^4\right)\right)-1 \\
&= \exp\left(\sum_{k\geq 1}C\left(\rho^k\right)\right)\exp\left(\left(C_2+a\right)X^2+C_3X^3+O\left(X^4\right)\right)-1 \\
&= \left(G(\rho)+1\right)\left(1+\left(C_2+a\right)X^2+C_3X^3+O\left(X^4\right)\right)-1,
\end{aligned}$$

with $a=\left[X^2\right]\sum_{k\geq 2}\frac{1}{k}C\left(\rho^k\left(1-X^2\right)^k\right)$. □

When we use the approximated values of ρ and σ from Section 3.1.3 we obtain

$$\begin{aligned}
G(x) \approx\ & 0.196549-0.295084X^2+0.0215044X^3 \\
&+0.118285X^4-0.0123625X^5-0.0482951X^6+0.0108644X^7.
\end{aligned}$$

Hence,

$$g_n \sim \frac{G_3}{\Gamma\left(-\frac{3}{2}\right)}n^{-\frac{5}{2}}\rho^{-n} \approx 0.00909941\, n^{-\frac{5}{2}}7.50360^n.$$

Table 3.4 shows the asymptotic estimates of the exact numbers that can be computed this way. We used Mathematica again.

3.2.2 Typical properties of a random unlabelled outerplanar graph

For the rest of the section, we assume Conjecture 3.3 to be true.

Theorem 3.9 (probability of connectivity). *The probability for a random outerplanar graph to be connected (under uniform distribution) is asymptotically*

$$\frac{C_3}{G_3} \approx 0.845721.$$

Proof. The probability that an outerplanar graph on n vertices is connected is exactly $\frac{c_n}{g_n}$. Using the asymptotic estimates for c_n and g_n from the Theorems 3.5 and 3.8 yields

$$\frac{c_n}{g_n} \sim \frac{C_3}{G_3}.$$ □

Theorem 3.10 (expected number of components). *Let κ_n denote the number of components in a random outerplanar graph on n vertices. Then the expectation of κ_n is asymptotically*

$$\mathbb{E}\left[\kappa_n\right] \sim 1+\sum_{k\geq 1}C\left(\rho^k\right) \approx 1.17847.$$

n	$s=1$	$s=2$	$s=3$	g_n exact
1	0	0	0	1
2	0	0	2	2
3	1	2	0	4
4	2	3	13	10
5	6	9	17	25
6	26	37	50	80
7	127	167	203	277
8	653	824	940	1150
9	3538	4321	4758	5291
10	19913	23729	25535	26918
11	115461	134983	142990	145744
12	685877	789640	827188	828856
13	4156824	4725798	4910017	4872771
14	25621286	28823296	29761291	29395784
15	160209818	178630202	183557038	180857382
16	1014313867	1122297556	1148869016	1130700488
17	6491736667	7135183190	7281787242	7163245811
18	41945902094	45835255493	46660298692	45895629266
19	273331278755	297139390631	301863941625	296937363511
20	1794596586205	1941979912637	1969454532633	1937625709854
21	11862841341954	12784428807549	12946410448754	12739784808937
22	78898746539258	84713818025456	85680649129918	84331837321404
23	527673859045748	564667858381048	570503168710670	561647630439975
24	3547008662079210	3784115104767780	3819691077051302	3761221057579892
25	23953788441774176	25483858290731958	25702755427821773	25314597326376883

Table 3.4: Asymptotic estimates $\sum_{k=1}^{s}\binom{n+k-\frac{1}{2}}{n} G_{2k+1}\rho^{-n}$ for the numbers of outerplanar graphs (rounded to the closest nonnegative integer) and the exact numbers g_n for comparison (Theorems 3.7 and 3.8).

Proof. As observed in the proof of Theorem 3.8, the cycle index sum for outerplanar graphs with exactly k components is $Z(S_k)[Z(\mathcal{C})]$. Hence, the probability that an outerplanar graph on n vertices has exactly k components is given by

$$\mathbb{P}[\kappa_n = k] = \frac{[x^n] Z(S_k; C(x))}{g_n}$$

and the expected number of components by

$$\mathbb{E}[\kappa_n] = \frac{1}{g_n}[x^n]\sum_{k\geq 1} kZ(S_k; C(x)).$$

Recall the recursion formula (1.2) for $Z(S_k)$,

$$Z(S_k) = \frac{1}{k}\sum_{r=1}^{k} s_r Z(S_{k-r}).$$

Then, when we let $Z(S_k) := 0$ for $k < 0$, and by use of formula (1.1),

$$\sum_{k\geq 1} kZ(S_k) = \sum_{k\geq 1}\sum_{r\geq 1} s_r Z(S_{k-r}) = \sum_{r\geq 1} s_r \sum_{k\geq 1} Z(S_{k-r}) = \sum_{r\geq 1} s_r \exp\left(\sum_{k\geq 1}\frac{1}{k}s_k\right). \tag{3.12}$$

Therefore,

$$\mathbb{E}\left[\kappa_n\right] = \frac{1}{g_n}\left[x^n\right]\left\{(G(x)+1)\sum_{r\geq 1} C(x^r)\right\}.$$

Since we have

$$G(x)+1 = 1+G(\rho)+G_2X^2+(1+G(\rho))\,C_3X^3+O\left(X^4\right),$$

and

$$\begin{aligned}\sum_{r\geq 1} C(x^r) &= C(\rho)+C_2X^2+C_3X^3+O\left(X^4\right)+\sum_{r\geq 2} C\left(\rho^k\left(1-X^2\right)^k\right)\\ &= \sum_{r\geq 1} C(\rho^r)+\left\{C_2+\left[X^2\right]\sum_{r\geq 2} C\left(\rho^r\left(1-X^2\right)^r\right)\right\}X^2+C_3X^3+O\left(X^4\right),\end{aligned}$$

we get

$$[X]\left\{(G(x)+1)\sum_{r\geq 1} C(x^r)\right\} = 0$$

and

$$\left[X^3\right]\left\{(G(x)+1)\sum_{r\geq 1} C(x^r)\right\} = (1+G(\rho))\,C_3+(1+G(\rho))\,C_3\sum_{r\geq 1} C(\rho^r)$$

and therefore (with $g_n \sim [x^n]\,(1+G(\rho))\,C_3X^3$),

$$\mathbb{E}\left[\kappa_n\right] \sim \frac{[x^n]\,(1+G(\rho))\,C_3\left(1+\sum_{r\geq 1} C(\rho^r)\right)X^3}{[x^n]\,(1+G(\rho))\,C_3X^3} \sim 1+\sum_{r\geq 1} C(\rho^r). \qquad \square$$

Given a family $\mathcal{A}$ of connected outerplanar graphs, we can make the following statements about the probability that a random outerplanar graph has exactly k components in $\mathcal{A}$. Denote the number of graphs in $\mathcal{A}$ which have exactly n vertices by a_n, and let $A(x) := \sum a_n x^n$ be the ordinary generating function for $\mathcal{A}$.

Theorem 3.11. *Assume the radius of convergence α of $A(x)$ to be strictly larger than ρ, $\alpha > \rho$, that is, $\mathcal{A}$ is exponentially smaller than $\mathcal{C}$. Let $\kappa_n^{\mathcal{A}}$ denote the number of components in an outerplanar graph on n vertices belonging to $\mathcal{A}$. Then the probability of a random outerplanar graph to have exactly $k \geq 0$ components belonging to $\mathcal{A}$ is*

$$\mathbb{P}\left[\kappa_n^{\mathcal{A}} = k\right] \sim Z\left(S_k; A(\rho)\right)\exp\left(-\sum_{r\geq 1}\frac{1}{r}A(\rho^r)\right)$$

and the expected number of components belonging to $\mathcal{A}$ in a random outerplanar graph is

$$\mathbb{E}\left[\kappa_n^{\mathcal{A}}\right] \sim \sum_{r\geq 1} A(\rho^r).$$

Proof. Outerplanar graphs with exactly $k \geq 0$ components in $\mathcal{A}$ are counted by

$$\begin{aligned}&Z\left(S_k; A(x)\right)\sum_{r\geq 0} Z\left(S_r; C(x)-A(x)\right)\\ =\ &Z\left(S_k; A(x)\right)\exp\left(\sum_{r\geq 1}\frac{1}{r}\left(C(x^r)-A(x^r)\right)\right)\\ =\ &Z\left(S_k; A(x)\right)\exp\left(-\sum_{r\geq 1}\frac{1}{r}A(x^r)\right)(G(x)+1).\end{aligned} \tag{3.13}$$

Since $A(x)$ is algebraic in ρ, the dominant singularity of this term is determined by $G(x)$. Thus

$$\begin{aligned}\mathbb{P}\left[\kappa_n^{\mathcal{A}} = k\right] &= \frac{[x^n]\, Z\left(S_k; A(x)\right) \exp\left(-\sum_{r\geq 1} \frac{1}{r} A\left(x^r\right)\right) \left(G(x)+1\right)}{[x^n]\, G(x)} \\ &\sim \frac{Z\left(S_k; A(\rho)\right) \exp\left(-\sum_{r\geq 1} \frac{1}{r} A\left(\rho^r\right)\right) [x^n]\, G(x)}{[x^n]\, G(x)}.\end{aligned}$$

For the expectation of $\kappa_n^{\mathcal{A}}$, observe that by Formula (3.12),

$$\sum_{k\geq 0} k Z\left(S_k; A(x)\right) = \exp\left(\sum_{r\geq 1} \frac{1}{r} A\left(x^r\right)\right) \sum_{r\geq 1} A\left(x^r\right).$$

Hence, by Formula (3.13), the expected number of components that belong to $\mathcal{A}$ becomes

$$\begin{aligned}\mathbb{E}\left[\kappa_n^{\mathcal{A}}\right] &= \frac{[x^n] \sum_k k Z\left(S_k; A(x)\right) \exp\left(-\sum_{r\geq 1} \frac{1}{r} A\left(x^r\right)\right) \left(G(x)+1\right)}{[x^n]\, G(x)} \\ &= \frac{[x^n] \exp\left(\sum_{r\geq 1} \frac{1}{r} A\left(x^r\right)\right) \sum_{r\geq 1} A\left(x^r\right) \exp\left(-\sum_{r\geq 1} \frac{1}{r} A\left(x^r\right)\right) \left(G(x)+1\right)}{[x^n]\, G(x)} \\ &\sim \sum_{r\geq 1} A\left(\rho^r\right).\end{aligned}$$

□

Corollary 3.12 (isolated vertices)**.** *The number of isolated vertices in a random outerplanar graph has asymptotically a geometric distribution with parameter ρ. The expected number of isolated vertices is asymptotically $\frac{\rho}{1-\rho} \approx 0.153761$.*

Proof. We let $\mathcal{A}$ to be the family consisting of the graph which is a single vertex, $A(x) = x$. Application of Theorem 3.11 yields

$$\begin{aligned}\mathbb{P}\left[\kappa_n^{\mathcal{A}} = k\right] &\sim Z\left(S_k; A(\rho)\right) \exp\left(-\sum_{r\geq 1} \frac{1}{r} A\left(\rho^r\right)\right) \\ &= \rho^k \exp\left(-\sum_{r\geq 1} \frac{1}{r} \rho^r\right) \\ &= \rho^k \exp\left(\log\left(1-\rho\right)\right) \\ &= \frac{\rho^k}{1-\rho}.\end{aligned}$$

□

Corollary 3.13 (two-connected components)**.** *The expected number of two-connected components in a random outerplanar graph is asymptotically*

$$\sum_{k\geq 1} D\left(\rho^k\right) \approx 0.175054.$$

Proof. We let $\mathcal{A} := \mathcal{D}$ be the family of dissections, $A(x) = D(x)$. The radius of convergence of $D(x)$ is $\delta = 3 - 2\sqrt{2} \approx 0.171573 > 0.136593 > \rho$. Application of Theorem 3.11 yields

$$\mathbb{E}\left[\kappa_n^{\mathcal{D}}\right] = \sum_{k\geq 1} D\left(\rho^k\right).$$

□

3.3 Chromatic number

Next to the limit laws for the number and type of components, we can also show that the chromatic number χ_n of a random unlabelled outerplanar graph on n vertices is asymptotically almost surely 3, i.e.,

$$\lim_{n\to\infty} \mathbb{P}[\chi_n = 3] = 1.$$

It is known that $\chi_n(G) \leq 3$ for all $G \in \mathcal{G}$ on n vertices (Start to colour one face with at most 3 colours and continue with neighbouring faces. Note that when doing so each uncoloured face has at most one coloured neighbour.). So it remains to show that $\mathbb{P}[\chi_n \leq 2] \to 0$, that is, we have to show that almost all outerplanar graphs are not bipartite (for $n \to \infty$). We do this by computation of an asymptotic estimate for the number of bipartite outerplanar graphs.

3.3.1 Counting formulas for bipartite outerplanar graphs

Observe that an outerplanar graph is bipartite if and only if all of its blocks are bipartite. As argumented in the introduction to Chapter 2, blocks of an outerplanar graph are dissections, and it is clear that a dissection is bipartite when all of its inner face have an even number of vertices. Hence, the computation of cycle index sums and generating functions for bipartite dissections, bipartite connected outerplanar graphs and bipartite outerplanar graphs works the same way as for the general case, except that we have to assure that every inner face has an even number of vertices.

We will use the same notation as before and add a small 'b' as subscript for indication that the investigated structures are now bipartite. Since the counting functions become more complex now, we work without closed formulas in the most cases from here on.

We start with the simplest case again, the oriented outer-edge rooted bipartite dissections.

Theorem 3.14 (bipartite oriented outer-edge rooted dissections). *The cycle index sum for bipartite oriented outer-edge rooted dissections is given by*

$$\begin{aligned} Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}\right) = \frac{s_1^2}{6} &+ \frac{\left(1-i\sqrt{3}\right)\left(6+s_1^2\right)s_1}{12\left(45s_1 - s_1^3 + 3\sqrt{3}\sqrt{-4s_1^4+71s_1^2-8}\right)^{1/3}} \\ &+ \frac{\left(1+i\sqrt{3}\right)s_1}{12}\left(45s_1 - s_1^3 + 3\sqrt{3}\sqrt{-4s_1^4+71s_1^2-8}\right)^{1/3}. \end{aligned} \tag{3.14}$$

Proof. Similarly to the proof of Theorem 2.1 (page 30) we take an oriented rooted $2m$-gon, $m \geq 2$, and plug $2m-1$ oriented outer-edge rooted bipartite dissections in the edges of the $2m$-gon other than the root edge. Summing over m and adding the single edge yields

$$Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}\right) = s_1^2 + s_1 \sum_{m\geq 2} \frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}\right)^{2m-1}}{s_1^{2m-1}} = s_1^2 + \frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}\right)^3}{s_1^2 - Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}\right)^2},$$

which is a cubic equation in $Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}\right)$. After solving this equation and picking the correct root, we obtain Formula (3.14). □

Theorem 3.15 (bipartite reflective outer-edge rooted dissections). *The cycle index sum for bipartite reflective outer-edge rooted dissections is given by*

$$Z\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right) = \frac{1}{2}\left(Z_{\mathrm{id}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right) + Z_{\mathrm{r}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right)\right),$$

where $Z_{\mathrm{id}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right)$ *counts only the identity mappings,*

$$Z_{\mathrm{id}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right) := 2Z\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right)|_{s_2=0} = s_1^2 \frac{s_1^2 - Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}; s_1^2\right)}{s_1^2 - 2Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}; s_1^2\right)}$$

and $Z_{\mathrm{r}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right)$ *counts the reflection mappings,*

$$Z_{\mathrm{r}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right) := 2Z\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right) - Z_{\mathrm{id}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right) = s_2 \frac{s_2 - Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}; s_2\right)}{s_2 - 2Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}; s_2\right)}.$$

Proof. Similarly to the proof of Theorem 2.3 (page 32) we take a rooted $2m+2$-gon, $m \geq 1$, and add $2m$ bipartite oriented outer-edge rooted dissections onto the edges of the $2m+2$-gon such that edges which have the same distance from the root edge receive the same dissection. At the edge opposite to the root edge we have to add a bipartite reflective outer-edge rooted dissection, see also Figure 2.3. Hence, we obtain

$$Z_{\mathrm{id}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right) = s_1^2 + \sum_{m\geq 1}\left(\frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}; s_1^2\right)}{s_1^2}\right)^m Z_{\mathrm{id}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right) = s_1^2 + \frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}; s_1^2\right)}{s_1^2 - Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}; s_1^2\right)} Z_{\mathrm{id}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right),$$

and similarly, we get

$$Z_{\mathrm{r}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right) = s_2 + \sum_{m\geq 1}\left(\frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}; s_2\right)}{s_2}\right)^m Z_{\mathrm{r}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right) = s_2 + \frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}; s_2\right)}{s_2 - Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}; s_2\right)} Z_{\mathrm{r}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right). \qquad \square$$

Theorem 3.16 (bipartite inner-edge rooted dissections)**.** *The cycle index sum for bipartite inner-edge rooted dissections is given by*

$$Z\left(\mathcal{E}_{\mathrm{i,b}}\right) = \frac{1}{4}\left(\left(\frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}\right)}{s_1} - s_1\right)^2 + \left(\frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}; s_2\right)}{s_2^2} - 1\right)\left(s_1^2 + s_2\right) + \left(\frac{Z_{\mathrm{r}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right)}{s_2} - 1\right)^2 s_2\right).$$

Proof. As in the proof of Theorem 2.5 (page 34), we have to consider the four kinds of transformations of the plane that map the root edge onto itself (see Figure 2.4):

1. The identity mapping
2. Reflection about the root-edge
3. Rotation about the root-edge
4. Reflection about the perpendicular bisector of the root-edge

Those dissections which are invariant under the identity mapping are counted by $\left(\frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}\right) - s_1^2}{s_1}\right)^2$, those which are invariant under reflections or rotations about the root edge are counted by $\frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}; s_2\right) - s_2^2}{s_2^2}\left(s_1^2 + s_2\right)$, and those dissections which are invariant under the fourth kind of mappings are counted by $\left(\frac{Z_{\mathrm{r}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}}\right) - s_2}{s_2}\right)^2 s_2$. $\square$

Theorem 3.17 (bipartite symmetry-edge rooted dissections)**.** *The cycle index sum for bipartite symmetry-edge rooted dissections is given by*

$$Z\left(\mathcal{E}_{\mathrm{s,b}}\right)=\frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}};s_1^2\right)}{2s_1^2}+\frac{s_1^2+s_2}{4s_2^2}Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}};s_2\right)-\frac{Z_{\mathrm{id}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}};s_1^2,s_2^2\right)}{4s_1^2}+\frac{Z_{\mathrm{r}}\left(\mathcal{E}^{\mathrm{r}}_{\mathrm{o,b}};s_1^2,s_2^2\right)}{4s_2}-\frac{s_1^2+s_2}{2}.$$

Proof. As in the proof of Theorem 2.7 (page 35), we construct a nonempty bipartite dissection rooted at a symmetry edge by taking a bipartite oriented outer-edge rooted dissection and joining it at the root-edge with a copy of itself. Doing so, we have to take a possible reflection of the chosen outer-edge rooted dissection into account, which yields the claimed formula. □

We proceed now with bipartite face rooted dissections. The main difference to the general case is here that the sum over the cycle indices of the cyclic groups affects only to groups of even order.

Theorem 3.18 (bipartite oriented face rooted dissections)**.** *The cycle index sum for bipartite oriented face rooted dissections is given by*

$$Z\left(\mathcal{F}^{\mathrm{o}}_{\mathrm{b}}\right)=-\frac{1}{2}\left(\frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}\right)}{s_1}\right)^2-\frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}};s_2\right)}{2s_2}-\sum_d\frac{\varphi(d)}{d}\begin{cases}\frac{1}{2}\log\left(1-\left(\frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}};s_d\right)}{s_d}\right)^2\right), & d \text{ odd},\\ \log\left(1-\frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}};s_d\right)}{s_d}\right), & d \text{ even}.\end{cases}$$

Proof. Similarly to the proof of Theorem 2.9 (page 36) consider an unrooted oriented k-gon, $k\geq 4$ even, with the cycle group C_k. A bipartite oriented face rooted dissection can be constructed by taking k bipartite oriented outer-edge rooted dissections and plugging them in the edges of the polygon, see Figure 2.5. By the Composition theorem 1.11, the cycle index sum for bipartite oriented dissections rooted at a face of size k is given by $Z\left(C_k\right)\left[\frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}\right)}{s_1}\right]$ and summing over k yields

$$Z\left(\mathcal{F}^{\mathrm{o}}_{\mathrm{b}}\right)=\sum_{k\geq 4\text{ even}}Z\left(C_k\right)\left[\frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}\right)}{s_1}\right]=\sum_{k\text{ even}}Z\left(C_k\right)\left[\frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}\right)}{s_1}\right]-Z\left(C_2\right)\left[\frac{Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}\right)}{s_1}\right].$$

Recall formula (1.3) for the cycle index of the cyclic group,

$$Z\left(C_k\right)=\frac{1}{k}\sum_{d\mid k}\varphi(d)\,s_d^{k/d}.$$

Then $Z\left(C_2\right)=\frac{1}{2}\left(s_1^2+s_2\right)$ and

$$\begin{aligned}\sum_{k\text{ even}}Z\left(C_k\right)&=\sum_{k\text{ even}}\frac{1}{k}\sum_{d\mid k}\varphi(d)\,s_d^{k/d}\\&=\sum_{\substack{m,d\\ md\text{ even}}}\frac{\varphi(d)}{md}s_d^m\\&=\sum_{d\text{ odd}}\frac{\varphi(d)}{d}\sum_{m\text{ even}}\frac{s_d^m}{m}+\sum_{d\text{ even}}\frac{\varphi(d)}{d}\sum_{m}\frac{s_d^m}{m}\\&=-\sum_{d\text{ odd}}\frac{\varphi(d)}{d}\frac{1}{2}\log\left(1-s_d^2\right)-\sum_{d\text{ even}}\frac{\varphi(d)}{d}\log\left(1-s_d\right).\end{aligned}$$ □

Theorem 3.19 (bipartite face rooted dissections). *The cycle index sum for bipartite face rooted dissections is given by*

$$Z(\mathcal{F}_{\mathrm{b}}) = \frac{1}{2} Z(\mathcal{F}_{\mathrm{b}}^{\mathrm{o}}) + \frac{1}{4s_2}\left(\frac{s_1^2}{s_2} Z(\mathcal{E}_{\mathrm{o,b}}^{\mathrm{o}}; s_2) + Z_{\mathrm{r}}(\mathcal{E}_{\mathrm{o,b}}^{\mathrm{r}})^2\right) \frac{Z\left(\mathcal{E}_{\mathrm{o,b}}^{\mathrm{o}}; s_2\right)}{s_2 - Z\left(\mathcal{E}_{\mathrm{o,b}}^{\mathrm{o}}; s_2\right)}.$$

Proof. As in the proof of Theorem 2.11 (page 38) we consider again an unrooted $2m+2$-gon, $m \geq 1$ even, together with the dihedral group D_{2m+2}. Recall formula (1.4) for the cycle index of the dihedral group,

$$Z(D_{2m+2}) = \frac{1}{2} Z(C_{2m+2}) + \frac{1}{4} s_2^{m+1} + \frac{1}{4} s_1^2 s_2^m.$$

Again we construct a bipartite face rooted dissection by plugging bipartite oriented outer-edge rooted dissections in the edges of the polygon. Those permutations which correspond to cycles are counted by $Z(C_{2m+2})$. For those permutations corresponding to reflections, we have to distinguish two cases here. The term s_2^{m+1} corresponds to $m+1$ bipartite oriented outer-edge rooted dissections which are attached to corresponding pairs of edges (Figure 2.7, left), and thus s_2^{m+1} has to be replaced by $\frac{s_1^2}{s_2}\left(\frac{Z(\mathcal{E}_{\mathrm{o,b}}^{\mathrm{o}};s_2)}{s_2}\right)^{m+1}$, where the term $\frac{s_1^2}{s_2}$ is due to the vertices of the root-face which are fixed by the reflection. The term $s_1^2 s_2^m$ corresponds to m bipartite oriented outer-edge rooted dissections which are attached to corresponding pairs of edges, and two bipartite outer-edge rooted reflective dissections (Figure 2.7, right), and thus $s_1^2 s_2^m$ has to be replaced by $\frac{Z_{\mathrm{r}}(\mathcal{E}_{\mathrm{o,b}}^{\mathrm{r}})^2}{s_2}\left(\frac{Z(\mathcal{E}_{\mathrm{o,b}}^{\mathrm{o}};s_2)}{s_2}\right)^m$. Hence we get

$$Z(\mathcal{F}_{\mathrm{b}}) = \sum_{m\geq 1} \frac{Z(C_{2m+2})}{2}\left[\frac{Z\left(\mathcal{E}_{\mathrm{o,b}}^{\mathrm{o}}\right)}{s_1}\right] + \frac{1}{4}\sum_{m\geq 1}\left(\frac{Z\left(\mathcal{E}_{\mathrm{o,b}}^{\mathrm{o}}; s_2\right)}{s_2}\right)^m \left(\frac{s_1^2}{s_2^2} Z(\mathcal{E}_{\mathrm{o,b}}^{\mathrm{o}}; s_2) + \frac{Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o,b}}^{\mathrm{r}}\right)^2}{s_2}\right).$$

□

Theorem 3.20. *The cycle index sum for bipartite face rooted dissections where the root-face is incident to a symmetry edge is given by*

$$Z(\mathcal{F}_{\mathrm{s,b}}) = \frac{Z\left(\mathcal{E}_{\mathrm{o,b}}^{\mathrm{o}}; s_1^2\right) - Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o,b}}^{\mathrm{r}}; s_1^2\right)}{s_1^2} + \frac{1}{2}\left(\frac{Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o,b}}^{\mathrm{r}}; s_1^2\right) - s_1^4}{s_1^2} + \frac{Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o,b}}^{\mathrm{r}}; s_1^2, s_2^2\right) - s_2^2}{s_2}\right).$$

Proof. As in the proof of Theorem 2.13 (page 39) we join two nonempty bipartite oriented outer-edge rooted dissections on their root-edge and declare one of the faces incident to the root edge to be the root face. Doing so, we have to take a possible reflection of the chosen outer-edge rooted dissection into account again. We obtain the claimed formula the same way as in the proof of Theorem 2.13. □

We are now ready to derive the cycle index sum for bipartite unrooted dissections.

Theorem 3.21 (bipartite dissections). *The cycle index sum for bipartite dissections is given by*

$$Z(\mathcal{D}_{\mathrm{b}}) = \frac{1}{2}\left(s_1^2 + s_2\right) + Z(\mathcal{F}_{\mathrm{b}}) - Z(\mathcal{F}_{\mathrm{s,b}}) - Z(\mathcal{E}_{\mathrm{i,b}}) + 2Z(\mathcal{E}_{\mathrm{s,b}}).$$

Proof. As in the proof of Theorem 2.14 (page 40) we see that the dual of a bipartite dissection is a tree. Thus we can apply Theorem 1.18, replacing $\mathcal{V}_1$ by $\mathcal{F}_{\mathrm{b}} \setminus \mathcal{F}_{\mathrm{s,b}}$, $\mathcal{E}$ by $\mathcal{E}_{\mathrm{i,b}}$, and $\mathcal{S}$ by $\mathcal{E}_{\mathrm{s,b}}$. Adding the single edge again yields the claimed Formula for $Z(\mathcal{D}_{\mathrm{b}})$. □

It remains to derive the cycle index sum for vertex rooted dissections.

Theorem 3.22 (bipartite vertex rooted dissections)**.** *The cycle index sum for bipartite vertex rooted dissections is given by*

$$Z\left(\mathcal{V}_{\mathrm{b}}\right) = s_1 \frac{\partial}{\partial s_1} Z\left(\mathcal{D}_{\mathrm{b}}\right).$$

Proof. We apply Theorem 1.14 with $\mathcal{G} = \mathcal{D}_{\mathrm{b}}$ and $\hat{\mathcal{G}} = \mathcal{V}_{\mathrm{b}}$. □

From the above cycle index sums, we can again obtain ordinary generating functions by application of Pólya's theorem, and exact numbers are then computed by development of these functions into power serieses. The exact numbers in Table 3.5 have been computed this way. Note that a dissection on an odd number of vertices cannot be bipartite, and thus we show only the numbers for even n here.

n	$(f_{\mathrm{b}})_n$	$(v_{\mathrm{b}})_n$	$(d_{\mathrm{b}})_n$
2	0	1	1
4	1	1	1
6	2	3	2
8	7	12	4
10	29	67	13
12	164	416	48
14	1033	2818	238
16	7184	19886	1325
18	52137	144881	8297
20	389851	1078822	54519

Table 3.5: Exact numbers for face rooted, vertex rooted, and unrooted bipartite dissections. (Theorems 3.19, 3.22, and 3.21)

We can now proceed to derive the cycle index sums for bipartite outerplanar graphs.

Theorem 3.23 (bipartite rooted connected outerplanar graphs)**.** *The cycle index sum for bipartite vertex rooted connected outerplanar graphs is implicitly determined by the equation*

$$Z\left(\hat{\mathcal{C}}_{\mathrm{b}}\right) = s_1 \exp\left(\sum_{k\geq 1} \frac{Z\left(\mathcal{V}_{\mathrm{b}}; Z\left(\hat{\mathcal{C}}_{\mathrm{b}}; s_k, s_{2k}\right), Z\left(\hat{\mathcal{C}}_{\mathrm{b}}; s_{2k}, s_{4k}\right), \ldots\right)}{k\, Z\left(\hat{\mathcal{C}}_{\mathrm{b}}; s_k, s_{2k}\right)}\right) \tag{3.15}$$

and the ordinary generating function satisfies

$$\hat{C}_{\mathrm{b}}(x) = x \exp\left(\sum_{k\geq 1} \frac{Z\left(\mathcal{V}_{\mathrm{b}}; \hat{C}_{\mathrm{b}}\left(x^k\right)\right)}{k\, \hat{C}_{\mathrm{b}}\left(x^k\right)}\right). \tag{3.16}$$

Proof. Formula (3.15) is an immediate consequence of Formula (1.7), page 17, for the cycle index sum of rooted connected graphs with given blocks. Formula (3.16) follows from Pólya's theorem by application of the mapping $s_k \mapsto x^k$. □

Theorem 3.24 (bipartite connected outerplanar graphs)**.** *The cycle index sum for bipartite connected outerplanar graphs is given by*

$$Z\left(\mathcal{C}_{\mathrm{b}}\right) = Z\left(\hat{\mathcal{C}}_{\mathrm{b}}\right) + Z\left(\mathcal{D}_{\mathrm{b}}; Z\left(\hat{\mathcal{C}}_{\mathrm{b}}\right)\right) - Z\left(\mathcal{V}_{\mathrm{b}}; Z\left(\hat{\mathcal{C}}_{\mathrm{b}}\right)\right) \tag{3.17}$$

and the ordinary generating function by

$$C_{\mathrm{b}}(x) = \hat{C}_{\mathrm{b}}(x) + Z\left(\mathcal{D}_{\mathrm{b}}; \hat{C}_{\mathrm{b}}(x)\right) - Z\left(\mathcal{V}_{\mathrm{b}}; \hat{C}_{\mathrm{b}}(x)\right). \tag{3.18}$$

Proof. Formula (3.17) is an immediate consequence of Theorem 1.16 about the unrooting of graphs with given blocks. □

Theorem 3.25 (bipartite outerplanar graphs). *The cycle index sum for bipartite outerplanar graphs is given by*

$$Z(\mathcal{G}_{\mathrm{b}}) = \left(\exp \sum_{k \geq 1} \frac{Z(\mathcal{C}_{\mathrm{b}}; s_k, s_{2k}, \ldots)}{k}\right) - 1$$

and the ordinary generating function by

$$G_{\mathrm{b}}(x) = \left(\exp \sum_{k \geq 1} \frac{1}{k} C_{\mathrm{b}}\left(x^k\right)\right) - 1. \tag{3.19}$$

Proof. A bipartite outerplanar graph is a collection of at least one bipartite connected outerplanar graph. Thus we obtain (by use of Formula (1.1))

$$Z(\mathcal{G}_{\mathrm{b}}) = \sum_{k \geq 1} Z(S_k)\left[Z(\mathcal{C}_{\mathrm{b}})\right] = \left(\exp \sum_{k \geq 1} \frac{Z(\mathcal{C}_{\mathrm{b}}; s_k, s_{2k}, \ldots)}{k}\right) - Z(S_0)\left[Z(\mathcal{C}_{\mathrm{b}})\right].$$

The ordinary generating function is given by Pólya's theorem. □

Exact numbers for bipartite vertex rooted outerplanar graphs can be computed from (3.16) the same way as for the general case (see Section 3.1.1.1). Having this done, the exact numbers for bipartite connected outerplanar graphs and bipartite outerplanar graphs are computed by power series expansion from (3.18) and (3.19). The exact numbers for bipartite outerplanar graphs with up to 12 vertices is given in Table 3.6.

n	$(\hat{c}_{\mathrm{b}})_n$	$(c_{\mathrm{b}})_n$	$(g_{\mathrm{b}})_n$
1	1	1	1
2	1	1	2
3	2	1	3
4	5	3	7
5	13	4	12
6	38	12	29
7	118	24	61
8	390	74	162
9	1348	193	412
10	4864	635	1221
11	18082	1990	3665
12	69023	6933	12031
13	268839	24144	40514
14	1064919	88203	143244
15	4275761	326053	518247
16	17363302	1234735	1926638
17	71177996	4732762	7285835
18	294135094	18403211	28013935
19	1223882843	72282202	109058879
20	5123054745	286705378	429366751

Table 3.6: Exact numbers for bipartite vertex rooted connected outerplanar graphs, bipartite connected outerplanar graphs, and bipartite outerplanar graphs. (Theorems 3.23, 3.24, and 3.25)

3.3.2 Asymptotics

Before we can compute asymptotic estimates for bipartite rooted connected outerplanar graphs, we have to determine the dominant singularity of the generating function of bipartite dissections. For convenience, we will not compute subexponential factors here.

Theorem 3.26 (exponential growth for bipartite dissections). *The number of bipartite oriented outer-edge rooted dissections on an even number n of vertices has the asymptotic estimate*

$$\left(e^o_{o,\mathrm{b}}\right)_n = \theta(n)\,\delta_{\mathrm{b}}^{-n}, \qquad \limsup_{\substack{n\to\infty\\ n\ \mathrm{even}}} (\theta(n))^{1/n} = 1,$$

with $\delta_{\mathrm{b}} := \sqrt{\frac{1}{8}\left(71 - 17\sqrt{17}\right)} \approx 0.336750$. *Further* $(f_{\mathrm{b}})_n$, $(v_{\mathrm{b}})_n$, *and* $(d_{\mathrm{b}})_n$ *have the same exponential growth* $\delta_{\mathrm{b}}^{-1} \approx 2.96956$ *(for even n).*

Proof. By Pólya's theorem we obtain from (3.14) the ordinary generating function

$$E^{\mathrm{o}}_{\mathrm{o,b}}(x) = Z\left(\mathcal{E}^{\mathrm{o}}_{\mathrm{o,b}}; x\right) = \frac{x^2}{6} + \frac{\left(1 - i\sqrt{3}\right)\left(6 + x^2\right)x}{12\left(45x - x^3 + 3\sqrt{3}\sqrt{-4x^4 + 71x^2 - 8}\right)^{1/3}} + \frac{\left(1 + i\sqrt{3}\right)x}{12}\left(45x - x^3 + 3\sqrt{3}\sqrt{-4x^4 + 71x^2 - 8}\right)^{1/3}.$$

Therefore, the singularities of $E^{\mathrm{o}}_{\mathrm{o,b}}(x)$ are determined by zeros of the term $-4x^4 + 71x^2 - 8$, which occur for $x^2 = \frac{1}{8}\left(71 \pm 17\sqrt{17}\right)$. Hence, by Theorem 1.28, and when we consider $E^{\mathrm{o}}_{\mathrm{o,b}}(x)$ as an function in x^2,

$$\left(e^o_{o,\mathrm{b}}\right)_n = [x^n]\,E^{\mathrm{o}}_{\mathrm{o,b}}(x) = \theta(n)\,\delta_{\mathrm{b}}^{-n}$$

for even n.

Since an unrooted dissection can only be rooted at at most n outer-edges and the Hamiltonian cycle has at most 2 possible orientations, we have the estimates

$$\frac{1}{2n}\left(e^o_{o,\mathrm{b}}\right)_n \le (d_{\mathrm{b}})_n \le \left(e^o_{o,\mathrm{b}}\right)_n.$$

Thus $(d_{\mathrm{b}})_n$ must have the same exponential growth rate δ_{b}^{-1}. Similarly, $(d_{\mathrm{b}})_n \le (v_{\mathrm{b}})_n \le n\,(d_{\mathrm{b}})_n$ and $(d_{\mathrm{b}})_n \le (f_{\mathrm{b}})_n \le n\,(d_{\mathrm{b}})_n$. ☐

To obtain an asymptotic expansion of $\hat{C}_{\mathrm{b}}(x)$ we proceed as in the general case. We conjecture that the dominant singularity of $\hat{C}_{\mathrm{b}}(x)$ is determined by a missing uniqueness when we try to invert (3.16) for x outside the disc of convergence of $\hat{C}_{\mathrm{b}}(x)$. Therefore, we let

$$H_{\mathrm{b}}(x,y) := x\exp\left(\frac{Z\left(\mathcal{V}_{\mathrm{b}}; y, \hat{C}_{\mathrm{b}}\left(x^2\right)\right)}{y} + \sum_{k\ge 2}\frac{Z\left(\mathcal{V}_{\mathrm{b}}; \hat{C}_{\mathrm{b}}\left(x^k\right), \hat{C}_{\mathrm{b}}\left(x^{2k}\right)\right)}{k\,\hat{C}_{\mathrm{b}}\left(x^k\right)}\right) - y$$

and apply Theorem 1.30. $H_{\mathrm{b}}(x,y)$ is analytic for all (x,y) with $\left|\hat{C}_{\mathrm{b}}\left(x^2\right)\right| < \delta_{\mathrm{b}}$ and $|y| < S_{\mathrm{b}} := \delta_{\mathrm{b}}$. Let $R_{\mathrm{b}} := \sqrt{\hat{C}_{\mathrm{b}}^{-1}(\delta_{\mathrm{b}})}$.

Conjecture 3.27. *The system*

$$H_{\mathrm{b}}(x,y) = 0, \qquad \frac{\partial}{\partial y}H_{\mathrm{b}}(x,y) = 0 \tag{3.20}$$

permits a solution $(\rho_{\mathrm{b}}, \sigma_{\mathrm{b}})$ with $\rho_{\mathrm{b}} \in (0, R_{\mathrm{b}})$ and $\sigma_{\mathrm{b}} \in (0, S_{\mathrm{b}})$. Furthermore, $\frac{\partial^2}{\partial y^2}H(\rho_{\mathrm{b}}, \sigma_{\mathrm{b}}) \neq 0$ *and* $\frac{\partial}{\partial x}H(\rho_{\mathrm{b}}, \sigma_{\mathrm{b}}) \neq 0$.

Now, the asymptotic expansion of $\hat{C}_{\mathrm{b}}(x)$ is an immediate consequence of Theorem 1.30.

Theorem 3.28 (asymptotic number of bipartite rooted connected outerplanar graphs)**.** *If Conjecture 3.27 is true, then the function $\hat{C}_{\mathrm{b}}(x)$ has an asymptotic expansion of the form*

$$\hat{C}_{\mathrm{b}}(x) = \sum_{k\geq 0} \left(\hat{C}_{\mathrm{b}}\right)_k X^k, \qquad X := \sqrt{1-\frac{x}{\rho_{\mathrm{b}}}}, \tag{3.21}$$

with

$$\left(\hat{C}_{\mathrm{b}}\right)_0 = \sigma_{\mathrm{b}}, \quad \left(\hat{C}_{\mathrm{b}}\right)_1 = -\sqrt{\frac{2\rho_{\mathrm{b}}\frac{\partial}{\partial x}H_{\mathrm{b}}(\rho_{\mathrm{b}},\sigma_{\mathrm{b}})}{\frac{\partial^2}{\partial y^2}H_{\mathrm{b}}(\rho_{\mathrm{b}},\sigma_{\mathrm{b}})}},$$

and constants $\left(\hat{C}_{\mathrm{b}}\right)_k$, $k\geq 2$, which can be computed from the derivatives of $H_{\mathrm{b}}(x,y)$ at $(\rho_{\mathrm{b}},\sigma_{\mathrm{b}})$. Furthermore, $(\hat{c}_{\mathrm{b}})_n$ has the asymptotic expansion

$$(\hat{c}_{\mathrm{b}})_n = \sum_{k\geq 0} \binom{n+k-\frac{1}{2}}{n} \left(\hat{C}_{\mathrm{b}}\right)_{2k+1} \rho_{\mathrm{b}}^{-n}$$

and particularly

$$(\hat{c}_{\mathrm{b}})_n \sim -\frac{\left(\hat{C}_{\mathrm{b}}\right)_1}{2\sqrt{\pi}} n^{-\frac{3}{2}} \rho_{\mathrm{b}}^{-n}.$$

The solution to (3.20) can be approximated similarly to Section 3.1.3. We replace $\hat{C}_{\mathrm{b}}(x)$ by a finite series $\hat{C}_{\mathrm{b}}^{[m]}(x) := \sum_{n=1}^{m} (\hat{c}_{\mathrm{b}})_n x^n$ and chop the infinite sum in $H_{\mathrm{b}}(x,y)$ at the same index m. That is, we consider the function

$$H_{\mathrm{b}}^{[m]}(x,y) := x\exp\left(\frac{Z\left(\mathcal{V}_{\mathrm{b}}; y, \hat{C}_{\mathrm{b}}^{[m]}(x^2)\right)}{y} + \sum_{k=2}^{m} \frac{Z\left(\mathcal{V}_{\mathrm{b}}; \hat{C}_{\mathrm{b}}^{[m]}(x^k), \hat{C}_{\mathrm{b}}^{[m]}(x^{2k})\right)}{k\,\hat{C}_{\mathrm{b}}(x^k)}\right) - y$$

and solve the system

$$H_{\mathrm{b}}^{[m]}(x,y) = 0, \quad \frac{\partial}{\partial y}H_{\mathrm{b}}^{[m]}(x,y) = 0. \tag{3.22}$$

Since the cycle index sums for the bipartite rooted dissections became very complex already, we abandon a reduction of the system by one variable, and solve the system (3.22) directly. For $m = 12$ we obtain the approximations

$$\rho_{\mathrm{b}} \approx 0.218475 \quad \text{and} \quad \sigma_{\mathrm{b}} \approx 0.333053$$

with residuals in the order of 10^{-96}. It is $R_{\mathrm{b}} \approx \sqrt{\left(\hat{C}_{\mathrm{b}}^{[m]}\right)^{-1}(\delta_{\mathrm{b}})} \approx 0.474712 > \rho_{\mathrm{b}}$ and $\sigma_{\mathrm{b}} < S_{\mathrm{b}} = \delta_{\mathrm{b}} \approx 0.336750$, which supports the correctness of Conjecture 3.27. Using this approximated values, we obtain (for $n\to\infty$)

$$(\hat{c}_{\mathrm{b}})_n \approx 0.023245\, n^{-\frac{3}{2}}\, 4.577174^n.$$

The expansion for the unrooted connected and not necessarily connected case follows as in the general case. We omit the proofs here.

Theorem 3.29 (asymptotic number of bipartite connected outerplanar graphs)**.** *If Conjecture 3.27 is true, then the function $C_{\mathrm{b}}(x)$ has an asymptotic expansion of the form*

$$C_{\mathrm{b}}(x) = C_{\mathrm{b}}(\rho_{\mathrm{b}}) + \sum_{k\geq 2} (C_{\mathrm{b}})_k X^k, \qquad X := \sqrt{1-\frac{x}{\rho_{\mathrm{b}}}}, \tag{3.23}$$

with constants $(C_{\mathrm{b}})_k$, $k \geq 2$, which can be computed from the constants $\left(\hat{C}_{\mathrm{b}}\right)_k$, and ρ_{b} is the same as in Theorem 3.28. Furthermore, $(c_{\mathrm{b}})_n$ has the asymptotic expansion

$$(c_{\mathrm{b}})_n = \sum_{k\geq 1} \binom{n+k-\frac{1}{2}}{n} (C_{\mathrm{b}})_{2k+1} \rho_{\mathrm{b}}^{-n}$$

and particularly

$$(c_{\mathrm{b}})_n \sim \frac{3\,(C_{\mathrm{b}})_3}{4\sqrt{\pi}} n^{-\frac{5}{2}} \rho_{\mathrm{b}}^{-n}.$$

Corollary 3.30 (asymptotic number of bipartite components). *If Conjectures 3.3 and 3.27 are true, then the expected number of bipartite components in an unlabelled outerplanar graphs is asymptotically*

$$\sum_{r\geq 1} C_{\mathrm{b}}(\rho^r) \approx 0.175427.$$

Proof. We apply Theorem 3.11 with $\mathcal{A} = \mathcal{C}_{\mathrm{b}}$. □

Theorem 3.31 (asymptotic number of bipartite outerplanar graphs). *If Conjecture 3.27 is true, then the function $G_{\mathrm{b}}(x)$ has an asymptotic expansion of the form*

$$G_{\mathrm{b}}(x) = G_{\mathrm{b}}(\rho_{\mathrm{b}}) + \sum_{k\geq 2} (G_{\mathrm{b}})_k X^k, \qquad X := \sqrt{1-\frac{x}{\rho_{\mathrm{b}}}},$$

with constants $(G_{\mathrm{b}})_k$, $k \geq 2$, which can be computed from the constants $(C_{\mathrm{b}})_k$, and ρ_{b} as in Theorem 3.28. Furthermore, $(g_{\mathrm{b}})_n$ has the asymptotic expansion

$$(g_{\mathrm{b}})_n = \sum_{k\geq 1} \binom{n+k-\frac{1}{2}}{n} (G_{\mathrm{b}})_{2k+1} \rho_{\mathrm{b}}^{-n}$$

and particularly

$$(g_{\mathrm{b}})_n \sim \frac{3\,(G_{\mathrm{b}})_3}{4\sqrt{\pi}} n^{-\frac{5}{2}} \rho_{\mathrm{b}}^{-n}.$$

3.3.3 Chromatic number of a random unlabelled outerplanar graph

We are now ready to state the main result of this section.

Theorem 3.32 (chromatic number). *If Conjectures 3.3 and 3.27 are true, then the chromatic number χ_n of a random unlabelled outerplanar graph on n vertices is asymptotically almost sure 3.*

Proof. It is known that $\chi_n \leq 3$ for all outerplanar graphs on n vertices. Furthermore,

$$\mathbb{P}[\chi_n \leq 2] = \frac{(g_{\mathrm{b}})_n}{g_n} \sim \frac{(G_{\mathrm{b}})_3}{G_3} \frac{\rho_{\mathrm{b}}^{-n}}{\rho^{-n}} \to 0 \quad (n \to \infty),$$

since $\rho_{\mathrm{b}} > \rho$. □

3.4 Number of edges

Up to now we have parameterized graphs only by their number of vertices, which allowed us to investigate the number of components, the chromatic number, and several other properties of a random outerplanar graph. Another interesting question is the number of edges (exact or in average) in an outerplanar graph. The previous decomposition technique can be adapted easily so that we can compute exact numbers for a given number of vertices and edges. Recall that for the construction of rooted connected graphs with given blocks (Section 1.2.4), the vertices of a block had to be substituted by rooted connected graphs, and thus the cycle index sums were necessary for the application of the composition theorem. Since we do not need to replace edges, we also do not need to consider edge-permutations which are induced by the automorphism group of the graph, i.e., a distinction between the orbits of edges in an unlabelled outerplanar graph is not necessary for our counting. Hence, to derive counting formulas for vertices and edges we add the formal variable y to the cycle index to indicate an edge. Thus, for a graph G on n vertices, m edges, and permutation group $\Gamma(G)$ which has the vertices as object set, we define

$$Z(G; s_1, s_2, \ldots; y) := Z(\Gamma(G); s_1, s_2, \ldots; y) := y^m \frac{1}{|\Gamma(G)|} \sum_{\alpha \in \Gamma(G)} \prod_{k=1}^{n} s_k^{j_k(\alpha)}$$

as the cycle index of G which includes edges into the counting. The resulting counting formulas are then a combination of a cycle index sum and a univariate generating function. Application of Pólya's theorem, substituting s_i by x^i, yields an ordinary generating function again.

3.4.1 Counting formulas

When we apply the decomposition technique for dissections and outerplanar graphs as in Chapter 2 and 3, but add the formal variable y to count edges, we obtain the following formulas. Since their derivation is analogous to the case where edges were not counted separately, we omit the proofs here, except for one example, the outer-edge rooted dissections, which we denote by $\mathcal{E}_{\mathrm{o}}$. Furthermore, we do not derive closed formulas in many cases, which are rather complex.

3.4.1.1 Dissections

Theorem 3.33 (outer-edge rooted dissections)**.** *The cycle index sum for oriented outer-edge rooted dissections, counting edges additionally, is given by*

$$Z(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}) = \frac{s_1}{2(1+y)} \left(s_1 y + 1 - \sqrt{(s_1 y - 1)^2 - 4 s_1 y^2} \right).$$

The cycle index sum for reflective outer-edge rooted dissections, counting edges additionally, is given by

$$\begin{aligned} Z(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}) = s_1 & \frac{(s_1 y - 1)\left(s_1^2 y^2 - 2 s_1 y^2 - 1\right) - (1 + s_1 y) \sqrt{\left(s_1^2 y^2 - 1\right)^2 - 4 s_1^2 y^4}}{4\left(s_1^2 y^3 + s_1^2 y + y - 1\right)} \\ & + \frac{s_2\left(2y^2 + s_2 y^3 - y\right) - y\left(s_2 y^2 + 2 s_2 y^3 - 1\right) - (s_1 + s_2 y)\sqrt{\left(s_2 y^2 - 1\right)^2 - 4 s_2 y^4}}{4\left(s_2 y^3 + s_2 y^2 + y - 1\right)}. \end{aligned}$$

The cycle index sum for outer-edge rooted dissections, counting edges additionally, is given by

$$\begin{aligned} Z(\mathcal{E}_{\mathrm{o}}) = & \frac{s_1}{4(1+y)} \left(s_1 y + 1 - \sqrt{(s_1 y - 1)^2 - 4 s_1 y^2} \right) \\ & + \frac{s_2\left(2y^2 + s_2 y^3 - y\right) - s_1\left(s_2 y^2 + 2 s_2 y^3 - 1\right) - (s_1 + s_2 y)\sqrt{\left(s_2 y^2 - 1\right)^2 - 4 s_2 y^4}}{4\left(s_2 y^3 + s_2 y^2 + y - 1\right)}. \end{aligned}$$

Proof. The decomposition of outer-edge rooted dissections is performed the same way as before. We only have to add the formal variable y for edges to the formulas. For oriented outer-edge rooted dissections, we obtain $s_1 y \left(\frac{Z(\mathcal{E}_\mathrm{o}^\mathrm{o})}{s_1}\right)^k$ for the cycle index sum of oriented outer-edge rooted dissections where the root-edge belongs to a face of size $k+1$. The extra factor y is for the root edge, the division and multiplication by s_1 due to the overlapping vertices of the attached oriented outer-edge rooted dissections. Hence, summing up and adding $s_1^2 y$ for the single edge,

$$Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}\right) = s_1^2 y + s_1 y \sum_{k\geq 2} \frac{Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}\right)^k}{s_1^k} = s_1^2 y + y \frac{Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}\right)^2}{s_1 - Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}\right)}, \tag{3.24}$$

which is again a quadratic equation in $Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}\right)$.

For reflective outer-edge rooted dissections we distinguish again whether the face with the root-edge has odd or even length. To construct graphs for the former case, take a rooted $2m+1$-gon and plug oriented outer-edge rooted dissections in the edges of the polygon other than the root edge, see Figure 2.2. The cycle index sum for the identity mapping yields

$$s_1 y \left(\frac{Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}; s_1^2; y^2\right)}{s_1^2}\right)^m$$

and for the reflection mappings we obtain

$$s_1 y \left(\frac{Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}; s_2; y^2\right)^m}{s_2}\right).$$

Similarly, for rooted faces of even length, take a $2m+2$-gon, plug oriented outer-edge rooted dissections in corresponding pairs of edges and a reflective outer-edge rooted dissection opposite to the root edge, see Figure 2.3. We obtain the cycle index sums

$$y Z_\mathrm{id}\left(\mathcal{E}_\mathrm{o}^\mathrm{r}\right) \left(\frac{Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}; s_1^2; y^2\right)}{s_1^2}\right)^m$$

and

$$y Z_\mathrm{r}\left(\mathcal{E}_\mathrm{o}^\mathrm{r}\right) \left(\frac{Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}; s_2; y^2\right)}{s_2}\right)^m$$

for the identity mappings and reflection mappings, respectively.

Hence, summing over $m \geq 1$, and adding the case of the single edge, we obtain

$$\begin{aligned} Z_\mathrm{id}\left(\mathcal{E}_\mathrm{o}^\mathrm{r}\right) &= s_1^2 y + y \sum_{m\geq 1} \left(\frac{Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}; s_1^2; y^2\right)}{s_1^2}\right)^m \left(s_1 + Z_\mathrm{id}\left(\mathcal{E}_\mathrm{o}^\mathrm{r}\right)\right) \\ &= s_1^2 y + y \frac{Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}; s_1^2; y^2\right)}{s_1^2 - Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}; s_1^2; y^2\right)} \left(s_1 + Z_\mathrm{id}\left(\mathcal{E}_\mathrm{o}^\mathrm{r}\right)\right) \end{aligned}$$

and

$$\begin{aligned} Z_\mathrm{r}\left(\mathcal{E}_\mathrm{o}^\mathrm{r}\right) &= s_2 y + y \sum_{m\geq 1} \left(\frac{Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}; s_2; y^2\right)}{s_2}\right)^m \left(s_1 + Z_\mathrm{r}\left(\mathcal{E}_\mathrm{o}^\mathrm{r}\right)\right) \\ &= s_2 y + y \frac{Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}; s_2; y^2\right)}{s_2 - Z\left(\mathcal{E}_\mathrm{o}^\mathrm{o}; s_2; y^2\right)} \left(s_1 + Z_\mathrm{r}\left(\mathcal{E}_\mathrm{o}^\mathrm{r}\right)\right). \end{aligned}$$

Thus it follows

$$Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)=s_1\frac{\left(s_1y-1\right)\left(s_1^2y^2-2s_1y^2-1\right)-\left(1+s_1y\right)\sqrt{\left(s_1^2y^2-1\right)^2-4s_1^2y^4}}{2\left(s_1^2y^3+s_1^2y^2+y-1\right)}$$

and

$$Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)=\frac{s_2\left(2y^2+s_2y^3-y\right)-s_1\left(s_2y^2+2s_2y^3-1\right)-\left(s_1+s_2y\right)\sqrt{\left(s_2y^2-1\right)^2-4s_2y^4}}{2\left(y+s_2y^2+s_2y^3-1\right)}.$$

Finally, we use $Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)=\frac{1}{2}\left(Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)+Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)\right)$.

For the cycle index sum of outer-edge rooted dissections, we distinguish between reflective and non-reflective outer-edge rooted dissections. The cycle index sum of the former is given by $Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)$, the cycle index sum of the latter by $\frac{1}{2}\left(Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\right)-Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)\right)$, since each non-reflective outer-edge rooted dissection $\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\setminus\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)$ is counted twice by $Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\right)$. Therefore we obtain

$$Z\left(\mathcal{E}_{\mathrm{o}}\right)=Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)+\frac{1}{2}\left(Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\right)-Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)\right)=\frac{1}{2}\left(Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\right)+Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)\right).\qquad\square$$

Corollary 3.34. *The bivariate ordinary generating function of oriented edge rooted dissections, reflective edge rooted dissections, and edge rooted dissections, is given by*

$$\begin{aligned}
E_{\mathrm{o}}^{\mathrm{o}}(x,y) &= \frac{x}{2(y+1)}\left(xy+1-\sqrt{(xy-1)^2-4xy^2}\right),\\
E_{\mathrm{o}}^{\mathrm{r}}(x,y) &= x\frac{(xy-1)\left(x^2y^2-2xy^2-1\right)-(1+xy)\sqrt{\left(x^2y^2-1\right)^2-4x^2y^4}}{4\left(x^2y^3+x^2y^2+y-1\right)},\\
E_{\mathrm{o}}(x,y) &= \frac{xy\left(1-x^2y^2(1+y)+x^3y^2(1+y)+x\left(y^2+y-1\right)\right)}{2(1+y)\left(x^2y^3+x^2y^2+y-1\right)}\\
&\quad -\frac{x}{4(y+1)}\sqrt{(xy-1)^2-4xy^2}-\frac{x(1+xy)}{8\left(x^2y^3+x^2y^2+y-1\right)}\sqrt{\left(x^2y^2-1\right)^2-4x^2y^4},
\end{aligned}$$

respectively. The number of oriented outer-edge rooted dissections on n vertices and m edges is

$$\left(e_{\mathrm{o}}^{\mathrm{o}}\right)_{n,m}=\frac{1}{m}\binom{m}{n-1}\binom{n-3}{m-n},\qquad n\geq 3,\ n\leq m\leq 2n-3.\tag{3.25}$$

Proof. The bivariate ordinary generating functions are obtained by an application of Pólya's theorem 1.9, replacing s_i by x^i, $i\geq 1$. The exact numbers for oriented edge rooted dissections can be obtained by application of Lagrange's Inversion Formula (see also [29]) to the functional equation (3.24). $\square$

Theorem 3.35 (inner-edge and symmetry-edge rooted dissections)**.** *The cycle index sum for inner-edge rooted dissections, counting edges additionally, is given by*

$$Z\left(\mathcal{E}_{\mathrm{i}}\right)=\frac{y}{4}\left(\left(\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}\right)}{s_1y}-s_1\right)^2+\left(\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}};s_2;y^2\right)}{s_2^2y^2}-1\right)\left(s_1^2+s_2\right)+\left(\frac{Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}\right)}{s_2y}-1\right)^2s_2\right).$$

The cycle index sum for symmetry-edge rooted dissections, counting edges additionally, is given by

$$Z\left(\mathcal{E}_{\mathrm{s}}\right)=\frac{Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}};s_1^2;y^2\right)}{2s_1^2y}+\frac{s_1^2+s_2}{4s_2^2y}Z\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}};s_2;y^2\right)-\frac{Z_{\mathrm{id}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}};s_1^2,s_2^2;y^2\right)}{4s_1^2y}+\frac{Z_{\mathrm{r}}\left(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}};s_1^2,s_2^2;y^2\right)}{4s_2y}-\frac{s_1^2+s_2}{2}y.$$

Theorem 3.36 (face rooted dissections). *The cycle index sum for oriented face rooted dissections, counting edges additionally, is given by*

$$Z(\mathcal{F}^{\mathrm{o}}) = -\left\{\sum_d \frac{\varphi(d)}{d}\log\left(1-\frac{Z(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_d; y^d)}{s_d}\right)\right\} - \frac{Z(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}})}{s_1} - \frac{1}{2}\left(\left(\frac{Z(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}})}{s_1}\right)^2 + \frac{Z(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2; y^2)}{s_2}\right).$$

The cycle index sum for face rooted dissections, counting edges additionally, is given by

$$Z(\mathcal{F}) = \frac{1}{2}Z(\mathcal{F}^{\mathrm{o}}) + \frac{1}{2s_2}\left(s_1 Z_{\mathrm{r}}(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}) + \frac{s_1^2}{2s_2}Z(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2; y^2) + \frac{1}{2}Z_{\mathrm{r}}(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}})^2\right)\frac{Z(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2; y^2)}{s_2 - Z(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_2; y^2)}.$$

The cycle index sum for face rooted dissections where the root-face is incident to a symmetry edge, counting edges additionally, is given by

$$Z(\mathcal{F}_{\mathrm{s}}) = \frac{1}{s_1^2 y}Z(\mathcal{E}_{\mathrm{o}}^{\mathrm{o}}; s_1^2; y^2) - \frac{1}{2s_1^2 y}Z_{\mathrm{id}}(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2; y^2) + \frac{1}{2s_2 y}Z_{\mathrm{r}}(\mathcal{E}_{\mathrm{o}}^{\mathrm{r}}; s_1^2, s_2^2; y^2) - \frac{y}{2}(s_1^2 + s_2).$$

Theorem 3.37 (dissections). *The cycle index sum for dissections, counting edges additionally, is given by*

$$Z(\mathcal{D}) = \frac{1}{2}(s_1^2 + s_2)y + Z(\mathcal{F}) - Z(\mathcal{F}_{\mathrm{s}}) - Z(\mathcal{E}_{\mathrm{i}}) + 2Z(\mathcal{E}_{\mathrm{s}}).$$

Theorem 3.38 (vertex rooted dissections). *The cycle index sum for vertex rooted dissections, counting edges additionally, is given by*

$$Z(\mathcal{V}) = s_1\frac{\partial}{\partial s_1}Z(\mathcal{D}; s_1, s_2, \ldots; y).$$

3.4.1.2 Outerplanar graphs

Theorem 3.39 (rooted connected outerplanar graphs). *The cycle index sum for vertex rooted connected outerplanar graphs, counting edges additionally, is implicitly determined by the equation*

$$Z\left(\hat{\mathcal{C}}\right) = s_1\exp\left(\sum_{k\geq 1}\frac{1}{k}\frac{Z\left(\mathcal{V}; Z\left(\hat{\mathcal{C}}; s_k, s_{2k}; y^k\right), Z\left(\hat{\mathcal{C}}; s_{2k}, s_{4k}; y^{2k}\right), \ldots; y^k\right)}{Z\left(\hat{\mathcal{C}}; s_k, s_{2k}; y^k\right)}\right)$$

and the bivariate ordinary generating function satisfies

$$\hat{C}(x,y) = x\exp\left(\sum_{k\geq 1}\frac{1}{k}\frac{Z\left(\mathcal{V}; \hat{C}(x^k, y^k); y^k\right)}{\hat{C}(x^k, y^k)}\right). \tag{3.26}$$

Theorem 3.40 (connected outerplanar graphs). *The cycle index sum for connected outerplanar graphs, counting edges additionally, is given by*

$$Z(\mathcal{C}) = Z\left(\hat{\mathcal{C}}\right) + Z\left(\mathcal{D}; Z\left(\hat{\mathcal{C}}\right)\right) - Z\left(\mathcal{V}; Z\left(\hat{\mathcal{C}}\right)\right)$$

and the bivariate ordinary generating function by

$$C(x,y) = \hat{C}(x,y) + Z\left(\mathcal{D}; \hat{C}(x,y)\right) - Z\left(\mathcal{V}; \hat{C}(x,y)\right).$$

Theorem 3.41 (outerplanar graphs). *The cycle index sum for outerplanar graphs, counting edges additionally, is given by*

$$Z(\mathcal{G}) = \left(\exp \sum_{k \geq 1} \frac{Z\left(\mathcal{C}; s_k, s_{2k}, \ldots; y^k\right)}{k}\right) - 1$$

and the bivariate ordinary generating function by

$$G(x, y) = \left(\exp \sum_{k \geq 1} \frac{1}{k} C\left(x^k, y^k\right)\right) - 1.$$

3.4.2 Exact numbers

By power series expansion of the derived ordinary generating functions we can obtain exact numbers for various types of dissections and outerplanar graphs on a given number of vertices and edges. We give here the exact numbers for edge rooted dissections (Table 3.7), vertex rooted dissections (Table 3.8), and unrooted dissections (Table 3.9) on up to 15 vertices and up to 27 edges. Furthermore, we provide the exact numbers for rooted connected outerplanar graphs (Table 3.10), unrooted connected outerplanar graphs (Table 3.11), and outerplanar graphs (Table 3.12) on up to 10 vertices and up to 17 edges. The computations were done by Mathematica.

$n \setminus m$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	1	3	3	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	1	5	11	7	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	1	8	30	43	22	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	1	11	62	151	166	66	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	1	15	117	417	749	646	217	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	1	19	197	967	2509	3506	2505	715	0	0	0
11	0	0	0	0	0	0	0	0	0	0	1	24	316	2014	7030	14031	16038	9731	2438	0
12	0	0	0	0	0	0	0	0	0	0	0	1	29	476	3834	17222	45881	74278	71611	37798
13	0	0	0	0	0	0	0	0	0	0	0	0	1	35	695	6850	38280	130013	278564	377973
14	0	0	0	0	0	0	0	0	0	0	0	0	0	1	41	975	11575	78600	329933	895460
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	48	1338	18745	151600	767633

$n \setminus m$	21	22	23	24	25	26	27	sum-up: $(e_o)_n$
2	0	0	0	0	0	0	0	1
3	0	0	0	0	0	0	0	1
4	0	0	0	0	0	0	0	2
5	0	0	0	0	0	0	0	7
6	0	0	0	0	0	0	0	24
7	0	0	0	0	0	0	0	104
8	0	0	0	0	0	0	0	457
9	0	0	0	0	0	0	0	2162
10	0	0	0	0	0	0	0	10419
11	0	0	0	0	0	0	0	51623
12	8398	0	0	0	0	0	0	259528
13	315002	146986	29414	0	0	0	0	1323813
14	1598913	1865402	1367926	572054	104006	0	0	6824886
15	2558514	5756171	8794126	8989324	5884044	2228766	371516	35521826

Table 3.7: Exact numbers for outer-edge rooted dissections $(e_o)_{n,m}$ on $n = 2, \ldots, 15$ vertices and $m = 1, \ldots, 27$ edges. (Corollary 3.34)

$n \setminus m$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	1	3	3	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	1	6	12	9	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	1	8	30	43	22	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	1	12	64	157	169	71	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	1	15	117	417	749	646	217	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	1	20	200	979	2521	3531	2514	729	0	0	0
11	0	0	0	0	0	0	0	0	0	0	1	24	316	2014	7030	14031	16038	9731	2438	0
12	0	0	0	0	0	0	0	0	0	0	0	1	30	480	3854	17252	45957	74332	71709	37826
13	0	0	0	0	0	0	0	0	0	0	0	0	1	35	695	6850	38280	130013	278564	377973
14	0	0	0	0	0	0	0	0	0	0	0	0	0	1	42	980	11605	78660	330113	895650
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	48	1338	18745	151600	767633

$n \setminus m$	21	22	23	24	25	26	27	sum-up: v_n
2	0	0	0	0	0	0	0	1
3	0	0	0	0	0	0	0	1
4	0	0	0	0	0	0	0	3
5	0	0	0	0	0	0	0	7
6	0	0	0	0	0	0	0	28
7	0	0	0	0	0	0	0	104
8	0	0	0	0	0	0	0	474
9	0	0	0	0	0	0	0	2162
10	0	0	0	0	0	0	0	10495
11	0	0	0	0	0	0	0	51623
12	8440	0	0	0	0	0	0	259881
13	315002	146986	29414	0	0	0	0	1323813
14	1599311	1865626	1368304	572144	104138	0	0	6826574
15	2558514	5756171	8794126	8989324	5884044	2228766	371516	35521826

Table 3.8: Exact numbers for vertex rooted dissections $v_{n,m}$ on $n = 2, \ldots, 15$ vertices and $m = 1, \ldots, 27$ edges. (Theorem 3.38)

$n \setminus m$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	1	2	3	3	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	1	2	6	7	4	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	1	3	11	24	24	12	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	1	3	17	51	89	74	27	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	1	4	26	109	265	371	259	82	0	0	0
11	0	0	0	0	0	0	0	0	0	0	1	4	36	194	660	1291	1478	891	228	0
12	0	0	0	0	0	0	0	0	0	0	0	1	5	50	345	1477	3891	6249	6044	3176
13	0	0	0	0	0	0	0	0	0	0	0	0	1	5	65	550	3000	10061	21524	29133
14	0	0	0	0	0	0	0	0	0	0	0	0	0	1	6	85	870	5710	23747	64183
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	6	106	1293	10228	51349

$n \setminus m$	21	22	23	24	25	26	27	sum-up: d_n
2	0	0	0	0	0	0	0	1
3	0	0	0	0	0	0	0	1
4	0	0	0	0	0	0	0	2
5	0	0	0	0	0	0	0	3
6	0	0	0	0	0	0	0	9
7	0	0	0	0	0	0	0	20
8	0	0	0	0	0	0	0	75
9	0	0	0	0	0	0	0	262
10	0	0	0	0	0	0	0	1117
11	0	0	0	0	0	0	0	4783
12	733	0	0	0	0	0	0	21971
13	24302	11326	2282	0	0	0	0	102249
14	114541	133464	98000	40942	7528	0	0	489077
15	170904	384035	586696	599516	392528	148646	24834	2370142

Table 3.9: Exact numbers for dissections $d_{n,m}$ on $n = 2, \ldots, 15$ vertices and $m = 1, \ldots, 27$ edges. (Theorem 3.37)

$n \setminus m$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	sum-up: $\hat{c}_n$
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
3	0	0	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3
4	0	0	0	4	4	2	0	0	0	0	0	0	0	0	0	0	0	0	10
5	0	0	0	0	9	15	13	3	0	0	0	0	0	0	0	0	0	0	40
6	0	0	0	0	0	20	50	66	36	9	0	0	0	0	0	0	0	0	181
7	0	0	0	0	0	0	48	164	295	264	125	22	0	0	0	0	0	0	918
8	0	0	0	0	0	0	0	115	520	1219	1545	1137	432	71	0	0	0	0	5039
9	0	0	0	0	0	0	0	0	286	1633	4780	7943	8050	4815	1589	217	0	0	29313
10	0	0	0	0	0	0	0	0	0	719	5061	18016	37399	48912	40413	20639	5885	729	177773

Table 3.10: Exact numbers for rooted connected outerplanar graphs $\hat{c}_{n,m}$ on $n = 1, \ldots, 10$ vertices and $m = 0, \ldots, 17$ edges. (Theorem 3.39)

$n \setminus m$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	sum-up: c_n
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
3	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
4	0	0	0	2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	5
5	0	0	0	0	3	5	4	1	0	0	0	0	0	0	0	0	0	0	13
6	0	0	0	0	0	6	13	16	8	3	0	0	0	0	0	0	0	0	46
7	0	0	0	0	0	0	11	33	55	47	22	4	0	0	0	0	0	0	172
8	0	0	0	0	0	0	0	23	89	194	230	168	61	12	0	0	0	0	777
9	0	0	0	0	0	0	0	0	47	240	652	1028	1011	587	191	27	0	0	3783
10	0	0	0	0	0	0	0	0	0	106	657	2188	4316	5483	4401	2223	618	82	20074

Table 3.11: Exact numbers for connected outerplanar graphs $c_{n,m}$ on $n = 1, \ldots, 10$ vertices and $m = 0, \ldots, 17$ edges. (Theorem 3.40)

$n \setminus m$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	sum-up: g_n
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
3	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4
4	1	1	2	3	2	1	0	0	0	0	0	0	0	0	0	0	0	0	10
5	1	1	2	4	6	6	4	1	0	0	0	0	0	0	0	0	0	0	25
6	1	1	2	5	9	15	19	17	8	3	0	0	0	0	0	0	0	0	80
7	1	1	2	5	10	21	39	57	65	50	22	4	0	0	0	0	0	0	277
8	1	1	2	5	11	24	54	107	184	259	257	172	61	12	0	0	0	0	1150
9	1	1	2	5	11	25	61	140	307	607	995	1239	1080	599	191	27	0	0	5291
10	1	1	2	5	11	26	64	157	390	934	2043	3849	5686	6187	4612	2250	618	82	26918

Table 3.12: Exact numbers for outerplanar graphs $g_{n,m}$ on $n = 1, \ldots, 10$ vertices and $m = 0, \ldots, 17$ edges. (Theorem 3.41)

3.4.3 Limit laws

3.4.3.1 Perturbation of singularities

As seen in Section 1.3, the analysis of the singularities of a univariate generating function $F(z)$ allows the derivation of asymptotic forms of the coefficients $f_n = [z^n] F(z)$ in a power series expansion of $F(z)$. When the generating function $F(z)$ is extended to a bivariate generating function $F(z,u) = \sum_{n,m} f_{n,m} z^n u^m$ by categorizing with a second parameter m the combinatorial structures we are interested in, crucial information on the distribution of such structures is accessible from $F(z,y)$ by letting u vary in some neighbourhood of 1, as we will show next. Our goal is to analyze the limiting distribution (if it exists) of a sequence of random variables X_n which arise from an integer valued combinatorial parameter ξ defined on a combinatorial class $\mathcal{F}$. This section is based on [16, Section IX.4].

Let X_n be the restriction of ξ on $\mathcal{F}_n$, i.e., all objects in $\mathcal{F}$ having size n, equipped with the uniform distribution. The *probability generating function* $p_n(u)$ of $\mathcal{F}$ and the random variable X_n is

$$p_n(u) = \sum_m \mathbb{P}[X_n = m] u^m = \frac{1}{f_n} \sum_m f_{n,m} u^m = \frac{1}{f_n} [z^n] F(z,u).$$

We will consider $F(z,u)$ not so much as an analytic function of two complex variables z and u, but rather as a collection of deformations of $F(z) \equiv F(z,1)$, when u varies in some neighbourhood of $u = 1$. Cauchy's coefficient formula (Corollary 1.25) gives

$$f_n(u) := [z^n] F(z,u) = \frac{1}{2\pi i} \int_\gamma \frac{F(z,u)}{z^{n+1}} dz.$$

Thus we can investigate the way the parameter u affects the analysis of singularities of $F(z,u)$ with respect to z when u is fixed (see Section 1.3). Studying the *perturbation of the singularities* of $f_n(u)$ around $u = 1$, quantified by $p_n(u) = \frac{f_n(u)}{f_n(1)}$, gives the limiting distribution of the random variables X_n. We will need the following definitions and theorems from probability theory.

Definition 3.42 (convergence in distribution)**.** Let Y be a random variable with continuous distribution function $\Psi(x)$, $\Psi(x) := \mathbb{P}[Y \le x]$. A sequence of random variables Y_n with distribution functions $\Psi_n(x)$ is said to *converge in distribution* to Y, if, pointwise for all x,

$$\lim_{n\to\infty} \Psi_n(x) = \Psi(x).$$

It can be shown that convergence in distribution is equivalent to the convergence of integral transforms of the random variables.

Definition 3.43 (Laplace transform)**.** The *Laplace transform* of a real-valued random variable Y with distribution function $\Psi(x)$, also called the *moment generating function*, is defined by

$$\lambda(s) := \mathbb{E}\left[e^{sY}\right] = \int_{-\infty}^{\infty} e^{sx} d\Psi(x).$$

Theorem 3.44 (continuity of integral transforms [16, Thm. IX.4])**.** *Let Y, Y_n be random variables with Laplace transforms $\lambda(s)$, $\lambda_n(s)$ that exist in a common interval $[-s_0, s_0]$ for some $s_0 > 0$. If, pointwise for all $s \in [-s_0, s_0]$,*

$$\lim_{n\to\infty} \lambda_n(s) = \lambda(s),$$

then the Y_n converge in distribution to Y.

We can now state the main theorem about the convergence of the distribution functions of the random variables X_n.

Theorem 3.45 (central law [16, Thm. IX.7]). *Let X_n be nonnegative discrete random variables with probability generating function $p_n(u)$. Assume that, uniformly in a fixed complex neighbourhood of $u = 1$, for sequences $\beta_n, \kappa_n \to \infty$,*

$$p_n(u) = A(u)(B(u))^{\beta_n}\left(1 + O\left(\frac{1}{\kappa_n}\right)\right),$$

where $A(u)$, $B(u)$ are analytic at $u = 1$ and $A(1) = B(1) = 1$. Assume finally that $B(u)$ satisfies the so-called "variability condition",

$$B''(1) + B'(1) - B'(1)^2 \neq 0.$$

Under these conditions, the distribution of X_n is asymptotically Gaussian. The mean and variance of X_n satisfy

$$\begin{aligned}
\mathbb{E}[X_n] &= \beta_n B'(1) + A'(1) + O\left(\frac{1}{\kappa_n}\right), \\
\mathbb{V}[X_n] &= \beta_n\left(B''(1) + B'(1) - B'(1)^2\right) + \left(A''(1) + A'(1) - A'(1)^2\right) + O\left(\frac{1}{\kappa_n}\right).
\end{aligned}$$

Proof. First, we estimate the mean and variance of X_n, then we show that the Laplace transforms of the standardized (mean-shifted and scaled by variance) X_n converge pointwise to the Laplace transform of the standard Gaussian distribution. By Theorem 3.44, convergence of the distribution functions of X_n to a Gaussian distribution follows.

The Laplace transform of X_n is

$$\lambda_n(s) = \mathbb{E}\left[e^{sX_n}\right] = \sum_{k\geq 0} \mathbb{P}[X_n = k]\, e^{sk} = p_n(e^s).$$

Thus, by assumption, we obtain

$$\lambda_n(s) = A(e^s)(B(e^s))^{\beta_n}\left(1 + O\left(\frac{1}{\kappa_n}\right)\right)$$

uniformly in a complex neighbourhood of $u = 1$. We let

$$U(s) = \log A(e^s) \quad \text{and} \quad V(s) = \log B(e^s).$$

Thus,

$$\lambda_n(s) = e^{U(s)+\beta_n V(s)}\left(1 + O\left(\frac{1}{\kappa_n}\right)\right).$$

The function $\log \lambda_n(s)$ is analytic at 0 and since $1 + x \sim e^x$ for $x = o(1)$, it satisfies

$$\log \lambda_n(s) = U(s) + \beta_n V(s) + O\left(\frac{1}{\kappa_n}\right).$$

Because of analyticity, this asymptotic expansion carries over to derivatives at 0 with same error terms. Thus

$$\begin{aligned}
\frac{\partial}{\partial s} \log \lambda_n(s) &= U'(s) + \beta_n V'(s) + O\left(\frac{1}{\kappa_n}\right), \\
\frac{\partial^2}{\partial s^2} \log \lambda_n(s) &= U''(s) + \beta_n V''(s) + O\left(\frac{1}{\kappa_n}\right).
\end{aligned}$$

Observe that

$$\begin{aligned}\lambda_n'(0) &= p_n'(e^s)|_{s=0} = \sum_{k\geq 0} k\mathbb{P}[X_n=k] = \mathbb{E}[X_n],\\ \lambda_n''(0) &= \sum_{k\geq 0} k^2\mathbb{P}[X_n=k] = \mathbb{E}\left[X_n^2\right].\end{aligned}$$

Hence, using $\lambda_n(0) = \mathbb{E}\left[e^0\right] = 1$,

$$\mathbb{E}[X_n] = \lambda_n'(0) = \lambda_n(s)\frac{\partial}{\partial s}\log\lambda_n(s)\bigg|_{s=0} = U'(0) + \beta_n V'(0) + O\left(\frac{1}{\kappa_n}\right)$$

and

$$\begin{aligned}\mathbb{V}[X_n] &= \mathbb{E}\left[X_n^2\right] - \mathbb{E}[X_n]^2 = \lambda_n''(0) - \lambda_n'(0)^2\\ &= \lambda_n(s)\left(\frac{\partial}{\partial s}\log\lambda_n(s)\right)^2 + \lambda_n(s)\frac{\partial^2}{\partial s^2}\log\lambda_n(s) - \left(\lambda_n(s)\frac{\partial}{\partial s}\log\lambda_n(s)\right)^2\bigg|_{s=0}\\ &= U''(0) + \beta_n V''(0) + O\left(\frac{1}{\kappa_n}\right).\end{aligned}$$

By $U'(0) = \frac{\partial}{\partial s}\log A(e^s)\big|_{s=0} = A'(1)$, $U''(0) = \frac{\partial^2}{\partial s^2}\log A(e^s)\Big|_{s=0} = A''(1) + A'(1) - A'(1)^2$, and similarly for $V'(0)$, the estimates for $\mathbb{E}[X_n]$ and $\mathbb{V}[X_n]$ follow.

Next we consider the standardized variable

$$X_n^* := \frac{X_n - \beta_n U'(0)}{\sqrt{\beta_n U''(0)}}$$

with Laplace transform

$$\lambda_n^*(s) = \mathbb{E}\left[e^{sX_n^*}\right],$$

which fulfills

$$\begin{aligned}\log\lambda_n^*(s) &= -\frac{\beta_n U'(0)}{\sqrt{\beta_n U''(0)}} + \log\lambda_n\left(\frac{s}{\sqrt{\beta_n U''(0)}}\right)\\ &= -\frac{\beta_n U'(0)}{\sqrt{\beta_n U''(0)}} + \beta_n U\left(\frac{s}{\sqrt{\beta_n U''(0)}}\right) + V\left(\frac{s}{\sqrt{\beta_n U''(0)}}\right) + O\left(\frac{1}{\kappa_n}\right).\end{aligned}$$

Local expansions of $U(s)$ and $V(s)$ around $s=0$ to third and first order, respectively, yield

$$\log\lambda_n^*(s) = \frac{s^2}{2} + O\left(\frac{|s| + |s|^3}{\sqrt{\beta_n}}\right) + O\left(\frac{1}{\kappa_n}\right)$$

for $|s| < \sqrt{\beta_n}$. Hence,

$$\lim_{n\to\infty}\log\lambda_n^*(s) \to \frac{s^2}{2}$$

for all s in a fixed neighbourhood of 0. Since $e^{s^2/2}$ is the Laplace transform of the standard Gaussian distribution, Theorem 3.44 yields the convergence of the distribution functions of the X_n to the Gaussian distribution. □

Corollary 3.46. *Let $F(x,y)$ be a bivariate generating function of a family of objects $\mathcal{F}$, where the second parameter is generated by a random nonnegative discrete random variable ξ on $\mathcal{F}$, i.e., $[x^n y^m] F(x,y) = |\{F \in \mathcal{F} \,||F| = n, \xi(F) = m\}|$. Assume that, in a fixed complex neighbourhood of $y = 1$, $F(x,y)$ provides a singular expansion of the form*

$$F(x,y) = \sum_{k \geq 0} F_k(y) \left(\sqrt{1 - \frac{x}{x_0(y)}}\right)^k \tag{3.27}$$

with $x_0(y)$ the dominant singularity of $F(x,y)$, and for the coefficients $F_k(y)$ there exists an $k_0 \in \mathbb{N}$ odd, such that $F_k(y) \equiv 0$, $0 < k < k_0$ odd, and $F_{k_0}(y) \neq 0$, for all y in the neighbourhood of 1. Assume $x_0(y)$ and $F_{k_0}(y)$ to be analytic at $y = 1$, and that $x_0(y)$ satisfies the "variability condition" $x_0''(1)x_0(1) + x_0'(1)x_0(1) - x_0'(1)^2 \neq 0$.

Let X_n be the restriction of ξ onto all objects in $\mathcal{F}$ of size n, denoted by $\mathcal{F}_n$. Under these conditions, the distribution of X_n is asymptotically Gaussian with mean

$$\mathbb{E}[X_n] = -\frac{x_0'(1)}{x_0(1)} n + \frac{F_{k_0}'(1)}{F_{k_0}(1)} + O\left(\frac{1}{n}\right)$$

and variance

$$\mathbb{V}[X_n] = \left(-\frac{x_0''(1)}{x_0(1)} - \frac{x_0'(1)}{x_0(1)} + \left(\frac{x_0'(1)}{x_0(1)}\right)^2\right) n + \left(\frac{F_{k_0}''(1)}{F_{k_0}(1)} + \frac{F_{k_0}'(1)}{F_{k_0}(1)} - \left(\frac{F_{k_0}'(1)}{F_{k_0}(1)}\right)^2\right) + O\left(\frac{1}{n}\right).$$

Proof. The probability generating function is given by

$$p_n(y) = \frac{1}{|\mathcal{F}_n|}[x^n] F(x,y) = \frac{[x^n] F(x,y)}{[x^n] F(x,1)}$$

and by Theorem 1.29 we obtain

$$p_n(y) = \frac{F_{k_0}(y)\, x_0(y)^{-n}}{F_{k_0}(1)\, x_0(1)^{-n}} \left(1 + O\left(\frac{1}{n}\right)\right)$$

uniformly in the neighbourhood of $y = 1$. We let $A(y) := \frac{F_{k_0}(y)}{F_{k_0}(1)}$ and $B(y) := \frac{x_0(1)}{x_0(y)}$. Hence,

$$p_n(y) = A(y) B(y)^n \left(1 + O\left(\frac{1}{n}\right)\right).$$

The derivatives of $B(y)$ satisfy

$$\begin{aligned} B'(y) &= -\frac{x_0'(y)}{x_0(y)^2} x_0(1), \\ B''(y) &= -\frac{x_0''(y)}{x(y)^2} x_0(1) + \frac{2x_0'(y)^2}{x_0(y)^3} x_0(1). \end{aligned}$$

Thus the variability condition

$$B''(1) + B'(1) - B'(1)^2 = -\frac{x_0''(1)}{x_0(1)} - \frac{x_0'(1)}{x_0(1)} + \left(\frac{x_0'(1)}{x_0(1)}\right)^2 \neq 0$$

holds for $B(y)$ by assumption. Therefore, we can apply Theorem 3.45 (with $\beta_n = \kappa_n = n$) which yields the statement. ∎

3.4.3.2 Limit distribution of the number of edges in a random outerplanar graph

Theorem 3.47 (dissections). *The distribution of the number of edges in a random two-connected outerplanar graph (dissection) is asymptotically Gaussian with mean* $\left(1+\frac{1}{\sqrt{2}}\right)n+O(1)\approx 1.70711n$ *and variance* $\frac{1}{4\sqrt{2}}n+O(1)\approx 0.176777n$.

Proof. Recall from Corollary 3.34 that

$$E_{\mathrm{o}}^{\mathrm{o}}(x,y)=\frac{x}{2(y+1)}\left(xy+1-\sqrt{(xy-1)^2-4xy^2}\right),$$

and the singularities of $E_{\mathrm{o}}^{\mathrm{o}}(x,y)$ are determined by the equation

$$(xy-1)^2-4xy^2=0.$$

Hence, by solving this equation for y close to 1, the dominant singularity $\delta(y)$ of $E_{\mathrm{o}}^{\mathrm{o}}(x,y)$ with respect to x is

$$\delta(y)=2+\frac{1}{y}-2\sqrt{1+\frac{1}{y}}.$$

With the same arguments as before, $\delta(y)$ is also the dominant singularity of the generating functions for vertex rooted and unrooted dissections, and the existence of singular expansions of the form (3.27) is clear from their generating functions. Furthermore,

$$\begin{aligned}\delta'(y) &= -\frac{1}{y^2}+\frac{1}{y^2\sqrt{1+\frac{1}{y}}},\\ \delta''(y) &= \frac{2}{y^3}+\frac{1}{2y^4}\left(\left(1+\frac{1}{y}\right)^{-3/2}-\frac{4y}{\sqrt{1+\frac{1}{y}}}\right),\end{aligned}$$

and

$$-\frac{\delta''(1)}{\delta(1)}-\frac{\delta'(1)}{\delta(1)}+\left(\frac{\delta'(1)}{\delta(1)}\right)^2=\frac{1}{4\sqrt{2}}\neq 0,$$

that is the variability condition on $\delta(y)$ is fulfilled. Finally, Corollary 3.46 yields the statement. □

Theorem 3.48 (outerplanar graphs). *If Conjecture 3.3 is true, then the distribution of the number of edges in a random outerplanar graph is asymptotically Gaussian with mean* $\approx 1.54894n$ *and variance* $\approx 0.227504n$. *The same holds for random connected outerplanar graphs.*

Proof. Recall from Formula (3.26) that the generating function $\hat{C}(x,y)$ of rooted connected outerplanar graphs is implicitly defined by

$$\hat{C}(x,y)=x\exp\left(\sum_{k\geq 1}\frac{Z\left(\mathcal{V};\hat{C}\left(x^k,y^k\right);y^k\right)}{k\,\hat{C}\left(x^k,y^k\right)}\right).$$

We assume again that $\hat{C}(x,y)$ ceases to be analytic (for fixed y) due to a missing uniqueness when solving this equation. Hence, we let

$$H(x,y,z):=x\exp\left(\frac{Z\left(\mathcal{V};z,\hat{C}\left(x^2,y^2\right);y\right)}{z}+\sum_{k\geq 2}\frac{Z\left(\mathcal{V};\hat{C}\left(x^k,y^k\right),\hat{C}\left(x^{2k},y^{2k}\right);y^k\right)}{k\,\hat{C}\left(x^k,y^k\right)}\right)-z$$

and search for a solution $(x,z)=(\rho(y),\sigma(y))$ of the system

$$H(x,y,z)=0, \qquad \frac{\partial}{\partial z}H(x,y,z)=0. \tag{3.28}$$

For $y=1$, a solution is ensured by Conjecture 3.3. For y close to 1 a (unique) solution of the system (3.28) is ensured by the implicit function theorem. Since $H(x,y,z)$ is smooth, it remains to verify that the determinant of the Jacobian of system (3.28) with respect to x and z,

$$\begin{pmatrix} \frac{\partial}{\partial x}H(x,y,z) & \frac{\partial}{\partial z}H(x,y,z) \\ \frac{\partial}{\partial x}\frac{\partial}{\partial z}H(x,y,z) & \frac{\partial}{\partial z}\frac{\partial}{\partial z}H(x,y,z) \end{pmatrix},$$

does not vanish at $(x,y,z)=(\rho(1),1,\sigma(1))$. This is clear, since from Conjecture 3.3 we have $\frac{\partial}{\partial z}H(\rho(1),1,\sigma(1))=0$, $\frac{\partial}{\partial x}H(\rho(1),1,\sigma(1))\neq 0$, and $\frac{\partial^2}{\partial z^2}H(\rho(1),1,\sigma(1))\neq 0$. Hence, we have analytic functions $\rho(y)$ and $\sigma(y)$ such that

$$H(\rho(y),y,\sigma(y))=0, \qquad \frac{\partial}{\partial z}H(\rho(y),y,\sigma(y))=0, \tag{3.29}$$

$\frac{\partial^2}{\partial z^2}H(\rho(y),y,\sigma(y))\neq 0$, and $\frac{\partial}{\partial x}H(\rho(y),y,\sigma(y))\neq 0$ for y close to 1. By Theorem 1.30 we obtain a singular expansion

$$\hat{C}(x,y)=\sum_{k\geq 0}\hat{C}_k(y)\left(\sqrt{1-\frac{x}{\rho(y)}}\right)^k \tag{3.30}$$

with coefficients $\hat{C}_k(y)$ analytic at $y=1$ and $\hat{C}_1(y)\neq 0$.

To find $\rho'(1)$ and $\rho''(1)$ we compute first and second derivatives of the equations in (3.29) with respect to y, so that we can express $\rho'(y)$ and $\rho''(y)$ in terms of $\rho(y)$, $\sigma(y)$ and partial derivatives of $H(x,y,z)$ at $(x,z)=(\rho(y),\sigma(y))$. Using the approximated values from Section 3.1.3, $\rho(1)\approx 0.133269$ and $\sigma(1)\approx 0.170756$, we obtain $\rho'(1)\approx -0.206426$, $\rho''(1)\approx 0.495849$, and

$$-\frac{\rho''(1)}{\rho(1)}-\frac{\rho'(1)}{\rho(1)}+\left(\frac{\rho'(1)}{\rho(1)}\right)^2\approx 0.227504\neq 0,$$

and thus also the variability condition of Corollary 3.46 is fulfilled. It follows that the distribution of the number of edges in a random rooted connected outerplanar graph is asymptotically Gaussian with mean

$$-\frac{\rho'(1)}{\rho(1)}n+O(1)\approx 1.54894n$$

and variance

$$\left(-\frac{\rho''(1)}{\rho(1)}-\frac{\rho'(1)}{\rho(1)}+\left(\frac{\rho'(1)}{\rho(1)}\right)^2\right)n+O(1)\approx 0.227504n.$$

For unrooted connected and unrooted not necessarily connected outerplanar graphs it remains to observe that their dominant singularities are determined by $\hat{C}(x,y)$, and thus they obey a singular expansion in the form of (3.30), but with different coefficients. $\square$

Chapter 4

Conclusions

4.1 Summary

We have shown how to decompose and enumerate unlabelled outerplanar graphs. Using Burnside's lemma (Lemma 1.5) the enumeration of unlabelled graphs can be formulated as the enumeration of automorphism groups [20]. A permutation group is thereby represented by a cycle index, which makes cycle index sums a powerful equivalence to the exponential generating functions in labelled counting. Using the Composition theorem (Theorem 1.11), the cycle index sums for rooted graphs can be derived by decomposition of the graphs into simpler (or the same) graphs, as in the labelled case, and also the cycle index sum of the unrooted structure can be obtained by integration of the cycle index sum of the rooted structure (Theorems 1.14 and 1.16). Another technique for the unrooting of rooted graphs is the dissimilarity characteristic theorem [20, page 55], which gives a relation between the numbers of dissimilar blocks and dissimilar vertices in an unlabelled graph. For trees (Theorem 1.17), this theorem can easily be applied to derive the ordinary generating function of unrooted trees from the one for rooted trees. We have shown that, after a small modification, it can also be used to derive the cycle index sum of unrooted trees (Theorem 1.18). We have applied this knowledge to dissections of a convex polygon, following Read's decomposition [29], to obtain the cycle index sums for rooted and unrooted two-connected outerplanar graphs, respectively (Theorems 2.14 and 2.16). Having these formulas in our hand, it has become an easy application of the Composition theorem to derive the counting formulas for connected and not necessarily connected outerplanar graphs (Theorems 3.1, 3.2, and 3.7). After transformation of the cycle index sums to ordinary generating functions we have computed exact numbers for several kinds of rooted and unrooted outerplanar graphs (Tables 2.2, 2.3, 3.1, and 3.4).

For the computation of asymptotic estimates, we have used singularity analysis [16]. Treating a generating function as a complex-valued analytic function, asymptotic estimates of their coefficients (in a power series expansion) can be obtained by having a look at the location and type of the dominant singularities (Theorems 1.28, 1.29, and 1.30). We have derived asymptotic estimates for various kinds of rooted and unrooted dissections, where the growth-constant and subexponential factors are given by analytic expressions (Corollaries 2.2, 2.15, and 2.17). For rooted connected outerplanar graphs, the location of the dominant singularity is much more difficult to find, but we have shown how it can be approximated (Section 3.1.3) and with it the growth constant for unlabelled outerplanar graphs (Theorems 3.4, 3.5, and 3.8).

Furthermore, with the knowledge about asymptotic estimates, we have investigated properties regarding the number and type of the components of a random outerplanar graph (when the number of vertices grows), e.g., the probability of connectedness (Theorem 3.9), the expected number of components (Theorem 3.10), and the distribution of the number of isolated vertices (Corollary 3.12). Investigating bipartite outerplanar graphs allows the study of the chromatic number of a random outerplanar graph. Since outerplanar graphs are always three-colourable, it remains to estimate the number of bipartite outerplanar graphs on a given number of vertices. Proceeding as in the general case, we have shown that their number is exponentially smaller than in the general case (Theorem 3.31). Thus we can conclude that the chromatic number of a random unlabelled outerplanar graph is asymptotically three (Theorem 3.32). Finally, when a second combinatorial parameter is introduced into the counting, yielding a bivariate generating function, information on its combinatorial distribution can be derived by letting the newly introduced variable vary and examining the thereby provoked perturbation of the dominant singularity. We have applied this technique to the number of edges in a random outerplanar graph to show that their limit distribution is Gaussian. The expectation and variance can be at least approximated (Theorems 3.47 and 3.48).

A summary of the estimated growth constants and other parameters for unlabelled outerplanar graphs is presented in Table 4.1. We also include the corresponding labelled quantities, cf. [3], for comparision. Observe that in the two-connected case the estimated quantities for the unlabelled and labelled structures do not differ, since their dominant singularity is determined by the same equation (compare Corollary 2.2 and the formula for $B(x)$ in [3]). However, for the general case, different values can be observed.

	dissections	outerplanar graphs	
	unlabelled or labelled	unlabelled	labelled
growth constant	$\delta^{-1} \approx 5.82843$	$\rho^{-1} \approx 7.50360$	$\lambda^{-1} \approx 7.32098$
$\mathbb{P}$ [connectivity]	1	0.845721	0.861667
$\mathbb{E}$ [nr. of components]	1	1.17847	-
distr. of nr. of isolated vertices	Dirac	Geom(ρ)	Po(λ)
$\mathbb{E}$ [nr. of isolated vertices]	0	0.153761	0.136593
chromatic number	3	3	3
distr. of nr. of edges	Gaussian	Gaussian	Gaussian
$\mathbb{E}$ [nr. of edges]	$1.70711n$	$1.54894n$	$1.56251n$
$\mathbb{V}$ [nr. of edges]	$0.176777n$	$0.227504n$	$0.223992n$

Table 4.1: Summary of growth constants, typical properties and limit laws for (random) unlabelled and labelled dissections and outerplanar graphs

4.2 Outlook

The obtained results show that the concept of the cycle index is a powerful tool for the enumeration of unlabelled structures, and we are confident that it can be applied to more complicated classes of unlabelled graphs as well. An interesting but already very complicated class of graphs is the set of unlabelled planar graphs. While three-connected unlabelled planar graphs are well studied [1, 17, 22, 23, 33], the two-connected unlabelled case seems to be very challenging. Recall that the enumeration of unlabelled outerplanar graphs is based on the fact that their two-connected components can be identified with dissections of a convex polygon [36], which can be uniquely embedded in the plane (when drawing the unique hamiltonian cycle at the outer face). For two-connected planar graphs, such a unique embedding is no longer possible, so that the derivation of the automorphism groups becomes much harder. A less difficult class of graphs is probably the class of series-parallel graphs.

A graph is series-parallel if it does not contain the complete graph K_4 as a minor. Since both, K_5 and $K_{3,3}$, contain a K_4 as a minor, series-parallel graphs are planar. Recall that a graph is outerplanar if it does not contain a K_4 or $K_{2,3}$ as a minor. Hence, outerplanar graphs are also series-parallel. Another characterization of series-parallel graphs, which also offers an access to the decomposition of these graphs, is the following. A connected graph is series-parallel if it can be obtained from a single edge by subdivision and duplication of edges. It is two-connected when it can be obtained from a double edge by subdivision and duplication of edges [3].

4.2.1 Sampling

Finally, an application of the derived formulas in a sampling algorithm for outerplanar graphs is desirable. For rooted connected outerplanar graphs, Bodirsky and Kang have presented a sampling algorithm which generates such graphs uniformly at random in polynomial time [7]. They also outlined an algorithm, based on rejection sampling, for unrooted connected outerplanar graphs, which generates such graphs uniformly at random in expected polynomial time. Whether a deterministic algorithm for the sampling of unrooted connected or not necessarily connected outerplanar graphs exists, is currently not known. A candidate to sample such graphs under a distribution which is, at least, approximately uniform are Boltzmann samplers [13]. They have the advantage that they work directly with the generating functions and do not rely on explicit (recursion) formulas for the number of investigated graphs of a given number of vertices. On the other hand, it is not possible to fix the size of the sampled graph in advance, i.e., a Boltzmann sampler draws uniformly at random an object of size n, when the size of its output is conditioned to be the fixed value n. But using an additional parameter, the expected size of the sampled object can be adjusted. Boltzmann samplers have successfully been implemented for the random generation of labelled planar graphs [18], but the application of Boltzmann samplers to sample unlabelled outerplanar graphs is complicated by the fact that we are not able to evaluate the generating function $\hat{C}(x)$ of rooted connected outerplanar graphs exactly.

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Subsequent additions

- The exact numbers for unlabelled vertex rooted outerplanar graphs ($\hat{c}_n$ in Table 3.1) have been verified by Tomii [32], using the recursive formulas from [7].
- The exact numbers c_n, g_n, $(d_{\mathrm{b}})_n$, $(c_{\mathrm{b}})_n$, and $(g_{\mathrm{b}})_n$ for small n are now also available in Sloanes encyclopedia of integer sequences [31] as sequences A111563, A111564, A111757, A111758, and A111759, respectively.
- Conjectures 3.3 and 3.27 have been proven by Éric Fusy lately. The following proposition is taken from the upcoming paper [8].

  Proposition. *The generating function $\hat{C}(x)$ satisfies the conditions of Theorem 1.30 with the function $H(x,y)$ from Equation (3.5) and $(r,s) = (\rho,\sigma)$, where ρ is the dominant singularity of $\hat{C}(x)$ and $\sigma := \lim_{x\to\rho^-} \hat{C}(x)$.*

  Proof. The conditions $H(0,0) = 0$ and $\frac{\partial}{\partial y}H(0,0) = -1$ are easily verified. The positivity conditions on the coefficients of $H(x,y)$ follow from the posivity of the coefficients of $Z(\mathcal{V})$. The analyticity domain of $H(x,y)$ is determined by the dominant singularities of $Z(\mathcal{V})$; that is, $H(x,y)$ is analytic for $|y| < \delta$ and x such that $|x^l| < \rho$ and $|\hat{C}(x^l)| < \delta$ for each $l \geq 2$. Since $\hat{C}(x)$ is strictly increasing for positive x and since $\rho < 1$, $|\hat{C}(x^l)| \leq |\hat{C}(x^2)|$ for all $l \geq 2$ and $|x| < \rho$. Therefore, $H(x,y)$ is analytic for $|x| < R := \min(\sqrt{\rho}, \sqrt{\hat{C}^{-1}(\delta)})$ and $|y| < S := \delta$.

  We show next that $\rho < R$ and $\sigma < S$.

  1. We show $\sigma \leq S = \delta$. Let $\tilde{H}(x,y) := H(x,y) + y$. $\tilde{H}(x,y)$ satisfies $\tilde{H}(x,\hat{C}(x)) = \hat{C}(x)$ and has the same domain of analyticity as $H(x,y)$. Assume $\sigma > \delta$. Then there exists $x_0 < \rho$ such that $\hat{C}(x_0) = \delta$. Observe that, if $|x| < x_0$ then $|\hat{C}(x^2)| \leq |\hat{C}(x)| < \hat{C}(x_0) = \delta$. Thus $(x,\hat{C}(x))$ is in the analyticity domain of $\tilde{H}(x,y)$, so that $\tilde{H}(x,\hat{C}(x)) = \hat{C}(x)$. By continuity we obtain $\tilde{H}(x_0,\hat{C}(x_0)) = \hat{C}(x_0)$. We have now the contradiction that $\hat{C}(x)$ is analytic at x_0 since $x_0 < \rho$, whereas $\tilde{H}(x,\hat{C}(x))$ is singular at x_0 because $\hat{C}(x_0) = \delta$.
  2. From 1. we know that $\sigma \leq \delta$, i.e. $\rho \leq \hat{C}^{-1}(\delta)$. Hence $R = \sqrt{\rho} > \rho$.
  3. It remains to prove that $\sigma < S$. Assume $\sigma = \delta$. Observe from Theorem 2.16 and (3.2) that
  $$\hat{C}(x) = x\exp(\Psi(\hat{C}(x)) + A(x))$$
  where $\Psi(y) = 1/8(1+y-\sqrt{1-6y+y^2})$ has a dominant singularity at $y = \delta$, and where $A(x)$ is a generating function with nonnegative coefficients and analytic for $|x| < \rho$. Hence, for $0 < x < \rho$,
  $$\hat{C}'(x) \geq \hat{C}'(x)\Psi'(\hat{C}(x))\hat{C}(x),$$
  so that $\Psi'(\hat{C}(x)) \leq 1/\hat{C}(x)$. Thus, $\Psi'(\hat{C}(x))$ is bounded when $x \to \rho^-$, which contradicts the fact that $\lim_{y\to\delta^-} \Psi'(y) = +\infty$.

Thus, $H(x,y)$ is analytic at (ρ,σ) and $H(\rho,\sigma)=0$ is satisfied. As pointed out before, the dominant singularity ρ of $\hat{C}(x)$ is determined either by a singularity in a component of Equation (3.2), or by a non-uniqueness in the definition of $\hat{C}(x)$ by Equation (3.2). The relation $\sigma<\delta$ excludes the first case, so that the singularity is caused by a non-uniqueness of the inversion. Hence, the derivative of $H(x,y)$ with respect to y has to vanish at $(x,y)=(\rho,\sigma)$, since otherwise the implicit function theorem ensures a (unique) analytic continuation of $\hat{C}(x)$ at $x=\rho$. Therefore, Equations (1.15) are satisfied for $(r,s)=(\rho,\sigma)$.

Furthermore, it is easily verified that

$$\frac{\partial^2}{\partial y^2}H(x,y)\bigg|_{(x,y)=(\rho,\sigma)}=\frac{1}{\sigma}+\frac{\partial^2}{\partial s_1^2}\frac{Z(\hat{\mathcal{C}},s_1,\hat{C}(\rho^2))}{s_1}\bigg|_{s_1=\sigma}=\frac{1}{\sigma}+\frac{\sigma}{(\sigma^2-6\sigma+1)^{3/2}},$$

and

$$\frac{\partial}{\partial x}H(x,y)\bigg|_{(x,y)=(\rho,\sigma)}=\sigma\left(\frac{1}{\rho}+\frac{\partial}{\partial x}\frac{Z\left(\mathcal{V};\sigma,\hat{C}\left(x^2\right)\right)}{\sigma}+\sum_{k\geq 2}\frac{Z\left(\mathcal{V};\hat{C}\left(x^k\right),\hat{C}\left(x^{2k}\right)\right)}{k\,\hat{C}\left(x^k\right)}\bigg|_{x=\rho}\right).$$

From $0<\sigma<\delta$ and the fact that the derivative in $\frac{\partial}{\partial x}H(\rho,\sigma)$ is a derivative of a formal power series with positive coefficients evaluated at $\rho>0$, it follows that both derivatives are strictly positive and hence do not vanish.

Finally, the aperiodicity of $\hat{C}(x)$ is easily seen from the fact that $\hat{c}_1\neq 0$ and $\hat{c}_2\neq 0$. $\square$

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