Technische
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# Stochastic Programming using Algebraic Modeling Languages

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# Agenda

- Short SP Introduction
- What Algebraic Modeling Languages offer
- Solving SPs: Decomposition Algorithms
- An Example
- Rapid Algorithm Development in GAMS

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# Introduction

- Starting with a deterministic problem:

$$\begin{aligned} \min \quad & g_0(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, \ i = 1, \cdots, m \\ & x \in X \subset \mathbb{R}^n \end{aligned}$$

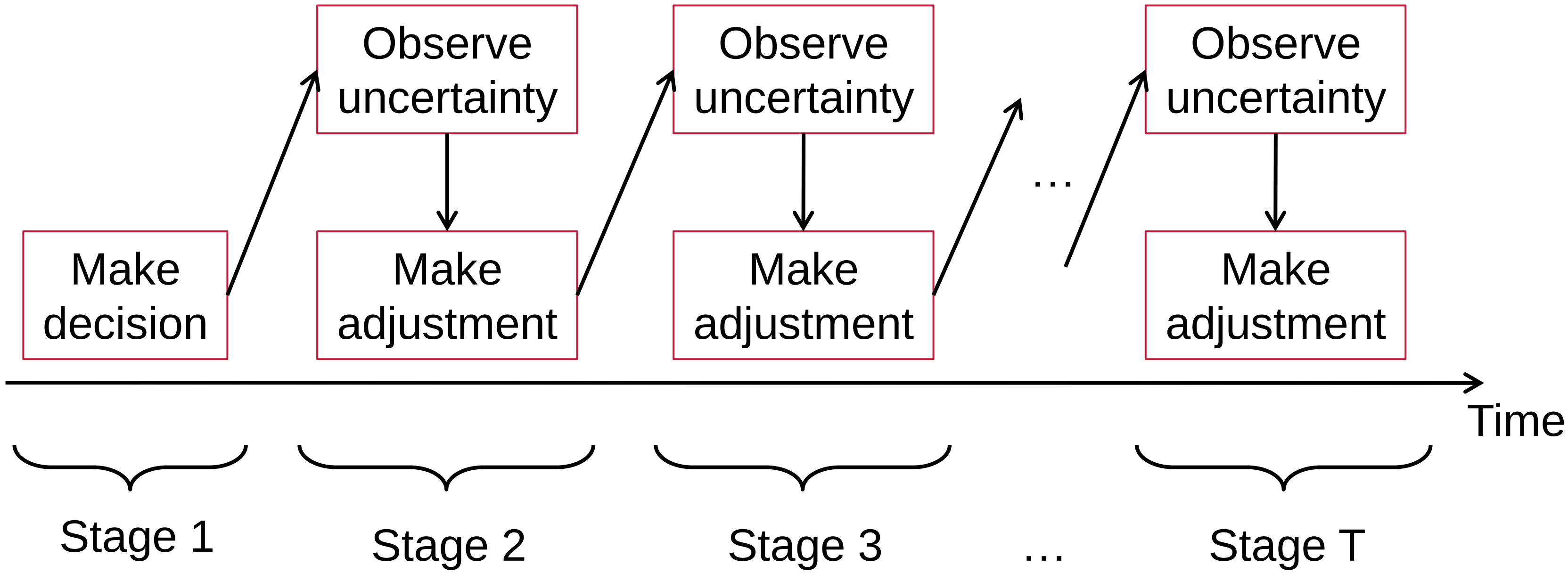
- Problem can be linear, nonlinear (convex, nonconvex) depending on the functions $g_i$ and the set $X$.
- Adding uncertainty:

$$\begin{aligned} \min \quad & g_0(x, \boldsymbol{\xi}) \\ \text{s.t.} \quad & g_i(x, \boldsymbol{\xi}) \leq 0, \ i = 1, \cdots, m \\ & x \in X \subset \mathbb{R}^n \end{aligned}$$

where $\boldsymbol{\xi}$ is a random vector varying over the set $\Xi \subset \mathbb{R}^k$

# Multi-stage decision making

- A stage is characterized by new information becoming known at the beginning of the stage and making recourse decisions / adjustments at the end of the stage (stage 1 is an exception):




- When making decisions only outcomes of the current stage and previous stages are available. For future stages only expectations exist.

  -> Non-anticipativity of the stochastic process

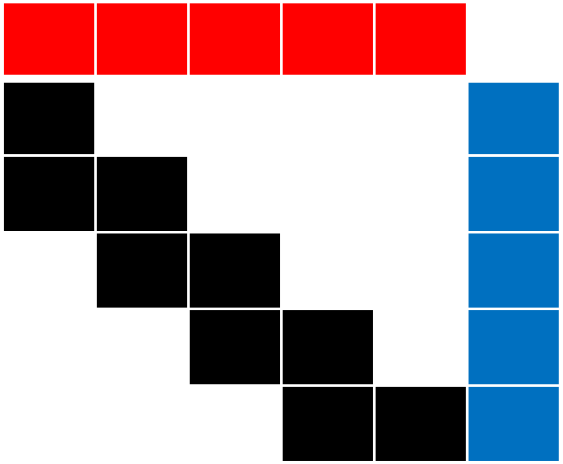
*Source: Fourer and Lopes (2009)*

# Multi-stage model

- It is often assumed that the decision in the current stage only depends on the previous stage. The multi-stage model then reads:

$$\min\ c_1x_1 + E[\min c_2\xi^2 x_2\xi^2 + \cdots + E[\min\ \ c_T(\xi^T)x_T(\xi^T)]\cdots]$$

$$\begin{aligned}\text{s.t.}\ \ & W_1x_1 = h_1 \\ & T_1(\xi^2)x_1 + W_2(\xi^2)x_2(\xi^2) = h_2(\xi^2) \\ & \quad \vdots \\ & T_{T-1}(\xi^T)x_{T-1}(\xi^{T-1}) + W_T(\xi^T)x_T(\xi^T) = h_T(\xi^T) \\ & x_1 \geq 0,\ \ x_t(\xi^t) \geq 0,\ \ t = 2, ..., T.\end{aligned}$$



- This assumption benefits decomposition algorithms because they can exploit the structure of the coeffecients matrix.
- Another common assumption is that the recourse matrices W are fixed (fixed recourse) or have a structure which eliminates the possibility of infeasibility (complete recourse).

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# Algebraic modeling languages

- Tested modeling languages:

# What algebraic modeling languages offer

- Features for stochastic programming:
  - Separation of deterministic core model and stochastic information
  - Ways of describing stochastic information:
    - Scenarios, either explicit provided by the user or constructed by the software (sampling techniques)
    - Continuous and discrete distributions (with intra- and interstage correlations)
  - Support for multi-stage and two-stage modeling, e.g. automatic assignment of variables and equations to stages
  - Scenario tree construction support, e.g. visualization and reduction methods
  - Decomposition algorithms, e.g. stochastic version of Benders‘ decomposition
  - Solution reports featuring
    - Solutions for each scenario
    - Value of the stochastic solution, expected value of perfect information
  - Reading from and writing into other formats, such as SMPS

# The most important features

- Features for stochastic programming:
  - **Separation** of deterministic core model and stochastic information
  - Ways of describing stochastic information:
    - **Scenarios**, either explicitly provided by the user or constructed by the software (**sampling techniques**)
    - Continuous and discrete distributions (with intra- and interstage **correlations**)
  - Support for multi-stage and two-stage modeling, e.g. **automatic assignment** of variables and equations to stages
  - Scenario tree construction support, e.g. visualization and reduction methods
  - **Decomposition algorithms**, e.g. stochastic version of Benders' decomposition
  - Solution reports featuring
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    - **Value of the stochastic solution**, expected value of perfect information
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# Basic idea

- Reconsider the multi-stage problem:

$$\min\ c_1x_1 + E[\min c_2\xi^2 x_2\xi^2 + \cdots + E[\min\ \ c_T(\xi^T)x_T(\xi^T)]\cdots]$$

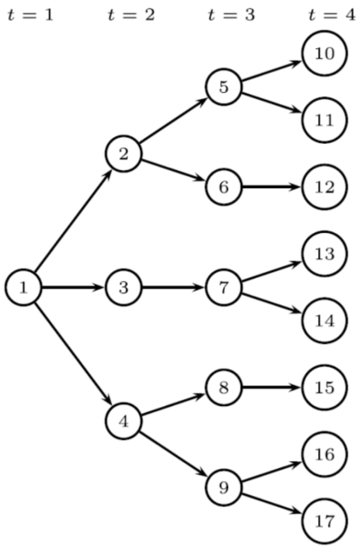
$$\text{s.t.}\ \ W_1x_1 = h_1$$

$$T_1(\xi^2)x_1 + W_2(\xi^2)x_2(\xi^2) = h_2(\xi^2)$$

$$\vdots$$

$$T_{T-1}(\xi^T)x_{T-1}(\xi^{T-1}) + W_T(\xi^T)x_T(\xi^T) = h_T(\xi^T)$$

$$x_1 \geq 0,\ \ x_t(\xi^t) \geq 0,\ \ t = 2, ..., T.$$




- With more stages and more scenarios the size explodes and the DE problem cannot be solved in a reasonable amount of time.
- Idea: Only look at the current stage and aggregate the future stages into a function depending on the current stage decision.

  -> The Future Cost Function

# One-stage subproblems

- Define one-stage dispatch subproblems:

$$\min\ c_t x_t + \hat{\alpha}_{t+1}(x_t)$$

$$\text{s.t.}\ \ W_t x_t \geq h_t - T_{t-1} x_{t-1}^*.$$

- Note that variables $x_{t-1}^*$ are fixed. Instead of solving the large multi-stage problem we solve a sequence of one-stage problems.
- Question: How to construct the future cost function(s) $\hat{\alpha}_{t+1}(x_t)$ ?
- Possible approaches:
  - SDP (Stochastic Dynamic Programming): A state-space approach using a recursion scheme and interpolations between obtained solution points
  - SDDP (Stochastic Dual Dynamic Programming) / Benders‘: Construct a piecewise linear function using supporting hyperplanes (cuts) through recursion.

# Construction of cuts (DDP)

- Solve the last stage subproblem (t=T):

$$\min\ c_t x_t + \hat{\alpha}_{t+1}(x_t)$$

$$\text{s.t. }\ W_t x_t \geq h_t - T_{t-1} x^*_{t-1}.$$

- Calculate marginals $\pi^1_T$ of the constraints and use them as dual multipliers for stage T-1:

$$\hat{\alpha}_{T-1}(x^*_{T-2}) = \min\ c_{T-1} x_{T-1} + \hat{\alpha}_T$$

$$\text{s.t. }\ W_{T-1} x_{T-1} \geq h_{T-1} - T_{T-2} x^*_{T-2}$$

$$\hat{\alpha}_T \geq \pi^1_T (h_T - T_{T-1} x_{T-1}).$$

- When calculating marginals for earlier stages the marginals for the cuts have to be accounted for as well (e.g. stage T-2):

$$\hat{\alpha}_{T-1} \geq \pi^1_{T-1}(h_{T-1} - T_{T-2} x_{T-2}) + \lambda^1_{T-1} \pi^1_{T-1} h_T.$$

*Source: Pereira and Pinto (1991)*

# Construction of cuts (DDP)

- The complete one-stage subproblem with cuts reads:

$$\left.\begin{aligned}
&\hat{\alpha}_t(x^*_{t-1}) = \min\ c_t^T x_t + \hat{\alpha}_{t+1} \\
&\qquad \text{s.t. } W_t x_t \geq h_t - T_{t-1} x^*_{t-1} \qquad \text{(1a)} \\
&\qquad\qquad \hat{\alpha}_{t+1} + \pi^j_{t+1} T_t x_t \geq \delta^j_t,\ j = 1, ..., J, \qquad \text{(1b)} \\
&\delta^j_t = \begin{cases} \pi^j_{t+1} h_{t+1}, & t = T-1 \\ \pi^j_{t+1} h_{t+1} + \sum_{i=1}^{j} \lambda^j_{i,t+1} \delta^i_{t+1}, & t = 1, ..., T-2, \end{cases}
\end{aligned}\right\} \text{(1)}$$

*Source: Pereira and Pinto (1991)*

# Construction of cuts (SDDP)

- Initialize:

  *Let $T$ be the planning horizon, initialize $\hat{\alpha}_{t+1}(x_t) = 0$ for $t = 1, ..., T$; iteration count J=0; Define a set of trial solutions $\{x^*_{tn},\ n = 1, ..., N,\ t = 1, ..., T\}$.*

- Carry out a Backward Recursion:

  *Backward recursion. Repeat for $t = T, ..., 2$ :*

  *Repeat for each trial decision $x^*_{t,n},\ n = 1, ..., N$:*

  *Repeat for each realization $h_{t,m},\ m = 1, ...M$:*

  *Solve problem (1) using trial decision $x^*_{t-1,n}$. Let $\pi^j_{t,m}$ and $\lambda^j_{i,t,m}$ be the multipliers associated to the constraints (1a) and (1b), respectively.*

  *Calculate the expected vertex value $\overline{\pi}^j_{t,n} = \sum_{m=1}^M p_{t,m}\pi^j_{t,m}$ and $\overline{\delta}^j_{t-1,n} = \sum_{m=1}^M p_{t,m}\delta^j_{t-1,m}$, and construct one supporting hyperplane of the approximate expected future cost function for stage $t-1$, $\overline{\alpha}_t(x_{t-1})$.*

- Solve the first-stage problem (1) for t=1, update $x^*_{1,n} := x^*_1$

*Source: Pereira and Pinto (1991)*

# Building a new solution and check for convergence (SDDP)

- Carry out a forward simulation:

*Forward Simulation. Repeat for* $t = 2, ..., T$*:*

*Repeat for* $n = 1, ..., N$*:*

*Sample a vector* $h_{tn}$ *from the set* $\{h_{tm},\ m = 1, ..., M\}$.

*Solve the two-stage subproblem (1) using trial decision* $x^*_{t-1,n}$*. Use the optimal solution to update* $x^*_{tn}$.

- Check for convergence:
  - Update the upper bound: $\overline{z} = c_1 x_1^* + \frac{1}{N}\sum_{n=1}^{N} z_n$
  - Update the lower bound: $\underline{z} = c_1 x_1^* + \overline{\alpha}_2(x_1)$.
  - If the difference between the upper confidence bound $\overline{z} + z_{q/2}\sigma_z/\sqrt{M}$ and the lower bound $\underline{z}$ is less than prescribed accuracy level: STOP

*Source: Pereira and Pinto (1991)*

*Source: Shapiro (2009)*

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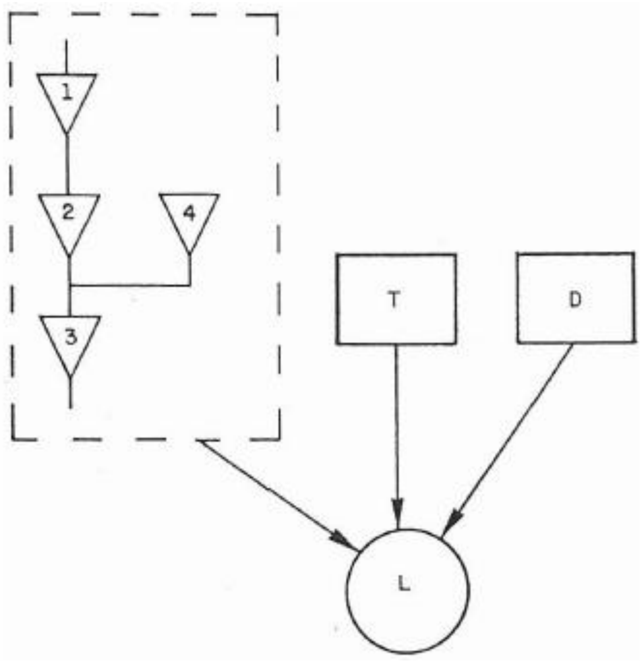
# The hydro power example

- 4 stages, right-hand side random (inflow), discrete distribution (64 scenarios)
- 4 reservoirs for hydro power generation
- Thermal power generation possible but expensive
- Demand L for power has to be satisfied (violation even more expensive)
- Water can be used in successive reservoirs.
- Decisions to make:
  - How much water should be stored and how much water should be used for power generation at each reservoir in each stage ?




*Source: Pereira and Pinto (1985)*

*Source: Velasquez, Restrepo, and Campo (1999)*

# The model formulation

$$\min \sum_{t=1}^{4} (c \cdot GT_t + f \cdot D_t)$$

$$\text{s.t.} \quad GT_t + \sum_{i=1}^{4} r_i Q_{i,t} + D_t = L \qquad \forall t$$

$$V_{i,t-1} + a_{i,t} + \sum_{j \in J_i} Q_{j,t} = V_{i,t} + Q_{i,t} \quad \forall i, t$$

$$Q_{i,t} \leq \overline{Q} \qquad \forall i, t$$

$$GT_t \leq \overline{GT} \qquad \forall t$$

$$GT_t,\ D_t,\ Q_{i,t},\ V_{i,t} \geq 0\ \forall i, t.$$

$GT_t$ Thermal power generation

$D_t$ Unsatisfied demand

$Q_{i,t}$ Water release for hydro power generation

$V_{i,t}$ Water volume stored in reservoir i at the end of stage t

*Source: Velasquez, Restrepo, and Campo (1999)*

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# SDDP in GAMS

- **Equations** can be used in multiple **Model** statements
  - Core model: `Model hydro /obj,cont,maxflow,therm,dem/;`
  - Submodels: `Model hydrosp / hydro, obj_approx, cuts /;`
- **Equation** generation controlled by **subsets**:

```
cont(i,tt(t)).. V(i,t-1) + V0(i)$sameas('t1',t) + A(i,t) - Q(i,t)
                   + sum(M(i,ii), Q(ii,t)) =e= V(i,t);
cuts(jj,n,tt(t))$docuts..
       ALPHA(t+1) - sum(i, cont_m(jj,n,i,t+1)*V(i,t)) =g= delta(jj,n,t);
```

- **Loops** over **sets**:
- Change **parameters** of a model

```
loop(s, // for all realizations
  A(i,tloop)=Astoch(s,i,tloop);
```

- Access of model stats for calculations:

```
delta(j,nloop,tloop-1)  =
        1/card(s)*[sum(i,-cont.m(i,tloop)*A(i,tloop)
        + maxflow.m(i,tloop)*Qcap(i))
        + therm.m(tloop)*GTcap + dem.m(tloop)*L
        + docuts*sum((jj,n), cuts.m(jj,n,tloop)*delta(jj,n,tloop))]
        + delta(j,nloop,tloop-1);
```

# Solver integration

- Running the SDDP algorithm with default GAMS settings is slow because „communicating“ with the solver for the small one-stage dispatch models takes up most of the time.
- „Communicating“ here means:
  - Model generation in GAMS
  - Writing the model to the hard drive (GAMS vacates memory)
  - Starting the solver
  - Restart GAMS; swap of GAMS database
- By using `hydrosp.solvelink = %solvelink.LoadLibrary%` the solver DLL is used in the GAMS process which means
  - GAMS stays in memory, no swap of GAMS database
  - Fast memory based model communication

# Scenario solver and comparison

- Another speed improvement is possible by using the GAMS scenario solver (beta state). Advantage: GAMS generates the model once and the scenario solver replaces the parameters accordingly for each solve.

```
Solve hydrosp min ACOST using LP scenario dict;
```

| Setting | Solve time (secs) |
|---|---|
| Solvelink=0 *(default)* | 40.297 |
| Solvelink=%Solvelink.LoadLibrary% | 03.625 |
| Scenario Solver | 00.797 |

(GAMS 23.5, lp=cplexd, Intel Core2 @ 2.0 GHz, 2GB)

- Using the scenario solver has to be done via the „Scenario Dict“