Stochastic Programming in GAMS

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2010

Example Model: Gas Price Model

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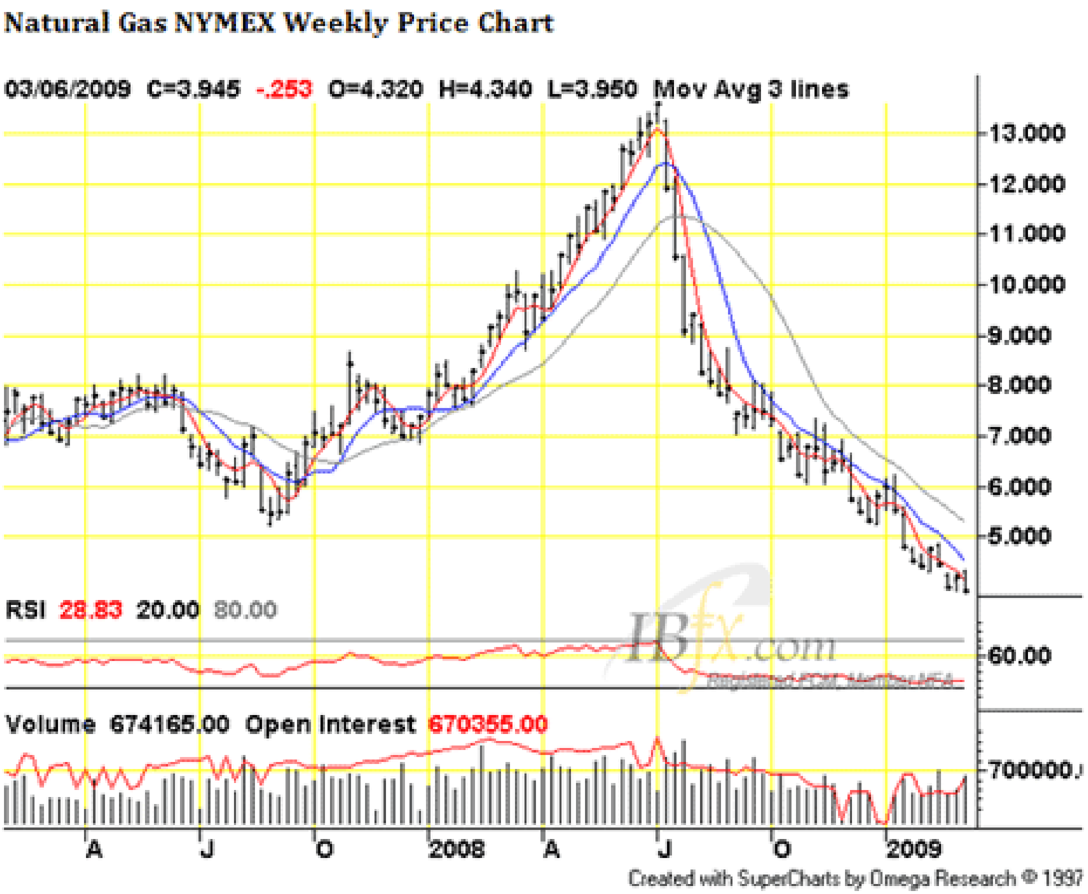

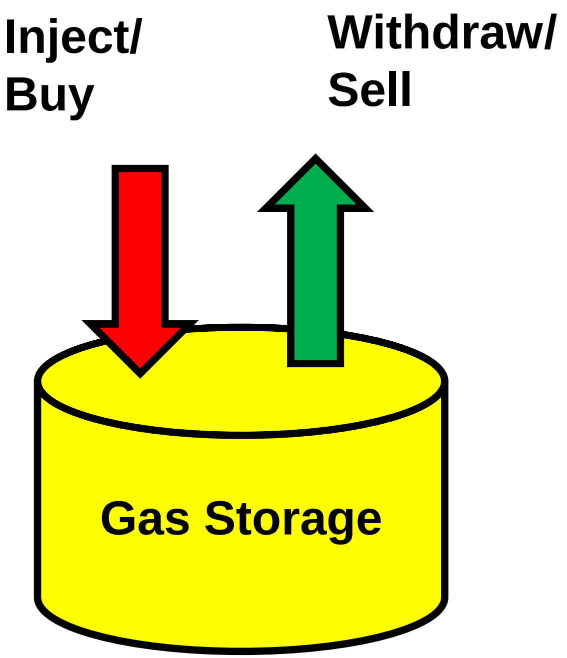

GAMS

n-Stage Stochastic Programs

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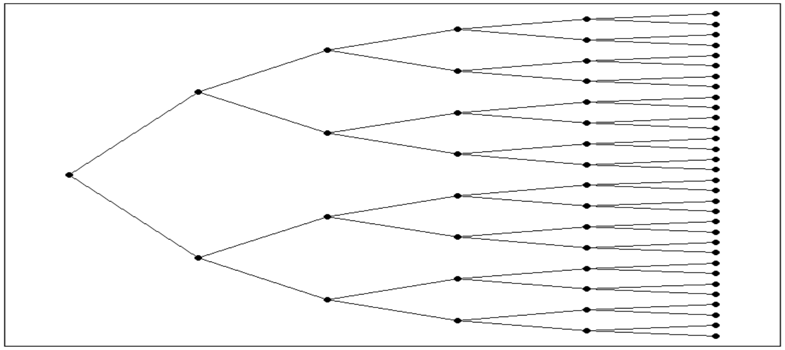
- Construct Scenario Tree:
  - Start with today's price and use a (discrete) distribution
  - Realizations: up, down
- Stochastic Linear Program (block structure)
  - Nested Bender's Decomposition (OSLSE, FortSP, AIMMS)
  - In practice Deterministic Equivalent with Barrier method

$$Z_{HN} = \min_{x_1} \left\{ c_1 x_1 + E_{\xi_2}\left[\min_{x_2} c_2 x_2 + E_{\xi_3|\xi_2}\left[\min_{x_3} c_3 x_3 + \ldots + E_{\xi_T|\xi_{T-1}|\ldots|\xi_2} \min_{x_T} c_T x_T \right]\right]\right\}$$

subject to:

$$\begin{aligned}
&A_{11}x_1 && && && &&= b_1\\
&A_{21}x_1 + && A_{22}x_2 && && &&= b_2\\
&A_{31}x_1 + && A_{32}x_2 + && A_{33}x_3 && &&= b_3\\
&\vdots && && && \ddots &&\vdots\\
&A_{T1}x_1 + && A_{T2}x_2 + && A_{T3}x_3 + && \ldots + A_{TT}x_T &&= b_T
\end{aligned}$$

$$\ell_t \le x_t \le u_t;$$



ScenRed (Römisch et. al., HU Berlin)

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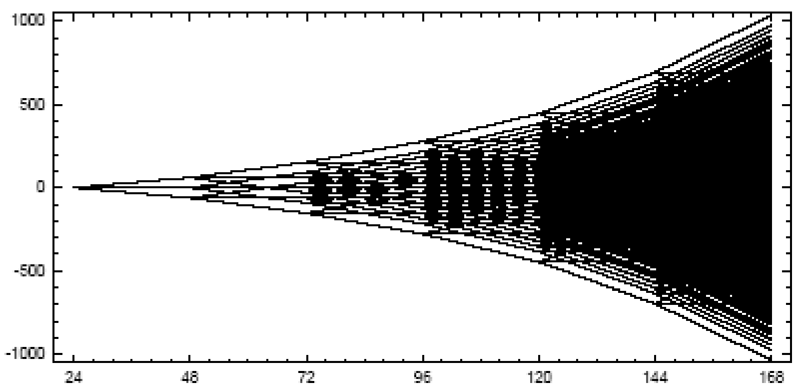

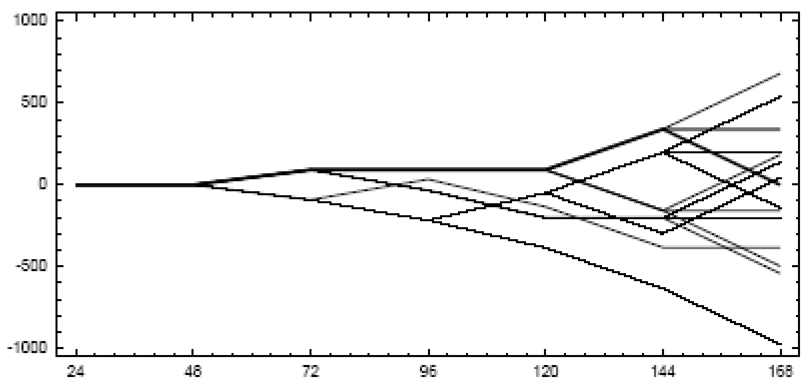

- Find good approximation of original scenario tree of significant smaller size
- Available since 2002
- Integrated in GAMS system
- No extra cost

Tree Generation: ScenRed2

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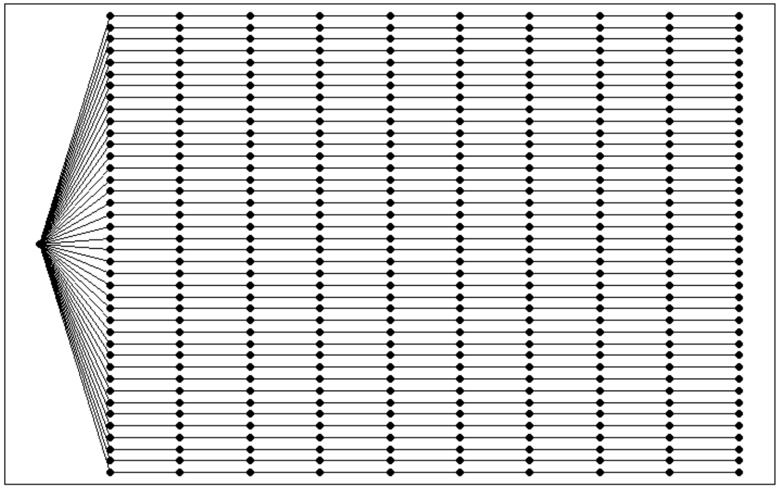
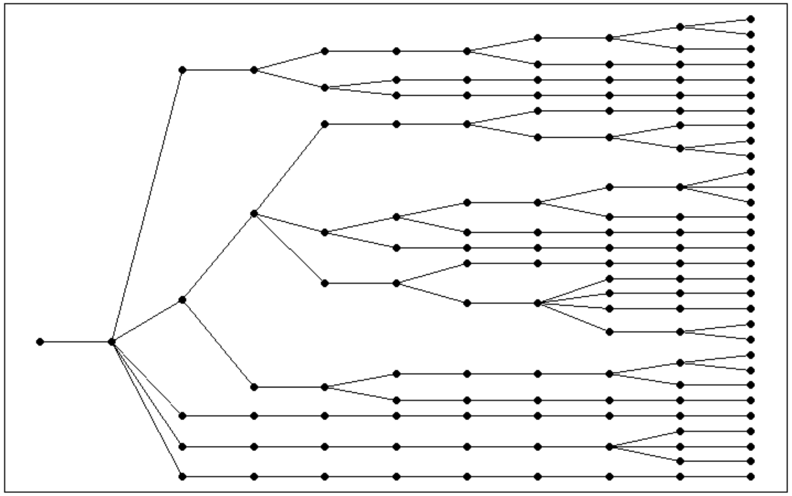
- Construct a true scenario tree from independent scenarios:





- Reconstruct underlying distribution from a set of scenarios

2-Stage Stochastic Programs

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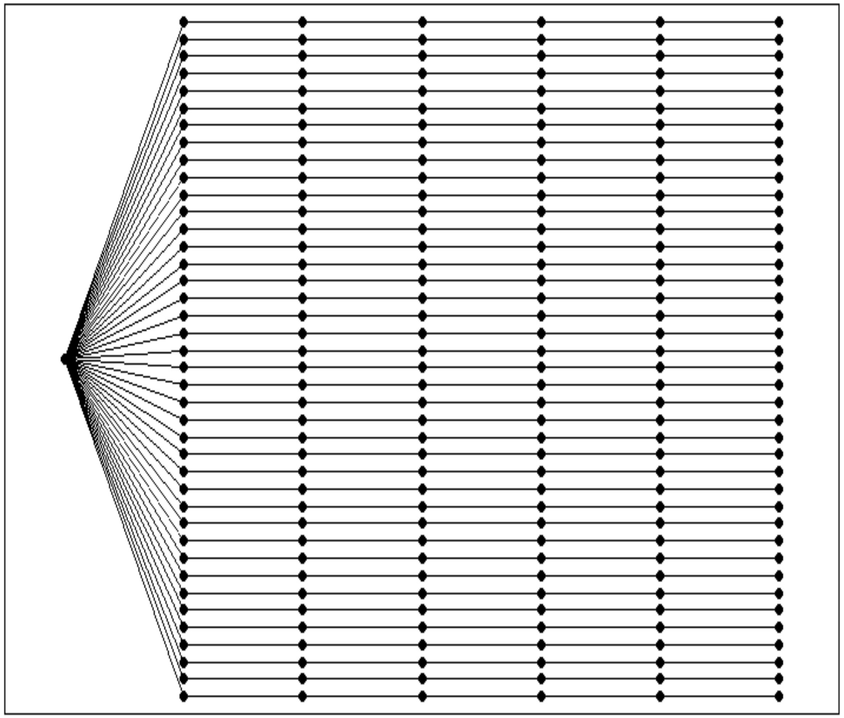
- SP Solver DECIS (Gerd Infanger, Stanford, USA)
  - Stores only one instance of the problem and generates scenario sub-problems as needed
  - Solution Strategies
    - Deterministic Equivalent (all scenarios)
    - Sampling:
      Crude Monte Carlo/
      Importance sampling



AML and Stochastic Programming (SP)

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- Algebraic Modeling Languages/Systems good way to represent optimization problems
  - Algebra is a universal language
  - Hassle free use of optimization solvers
  - Simple connection to data sources (DB, Spreadsheets, …) and analytic engines (GIS, Charting, …)
- Large number of (deterministic) models in production
  - Opportunity for *seamless* introduction of new technology like Global Optimization, Stochastic Programming, …
  - AML potential framework for SP

Simple Example

Simple Example: Newsboy (NB) Problem

- Data:
  - A newsboy faces a certain demand for newspapers
    $d = 45$
  - He can buy newspapers for fixed costs per unit
    $c = 30$
  - He can sell newspapers for a fixed price
    $r = 60$
  - For hold units he has to pay a disposal fee
    $h = 10$
  - He has to satisfy his customers demand or has to pay a penalty
    $p = 5$

- Decisions:
  - How many newspapers should he buy: X **45**
  - How many newspapers should he sell: S **45**

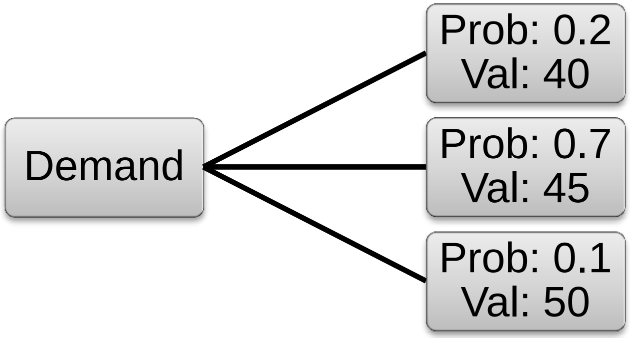
- Derived Outcomes:
  - How many newspapers need to be disposed: I **0**
  - How many customers are lost: L **0**

Simple NB Problem – GAMS Formulation

```
*             LostSales = demand - UnitsSold
lSales..      L =e= d - S;

*             Inventory = UnitsBought - UnitsSold
Inv..         I =e= X - S;

*             Profit, to be maximized
Profit..      Z =e= r*S - c*X - h*I - p*L;

Model nb / all /;

solve nb max z use lp;
```

NB Problem – Add Uncertainty

- Uncertain demand *d*



- Decisions to make:
  - How much newspaper should he buy “here and now” (without knowing the outcome of the uncertain demand)?
    → *First-stage decision*
  - How many customers are lost after the outcome becomes known?
    → *Second-stage* or *recourse decision*
  - Recourse decisions can be seen as
    - penalties for bad first-stage decisions
    - variables to keep the problem feasible

Stochastic NB Problem – GAMS Extension

```
* Make d uncertain
randvar d discrete  0.2 40
                    0.7 45
                    0.1 50

* Define non-default stages
stage 2 d I L S
stage 2 lSales Inv
```

New GAMS (EMP) Keywords

Excursus: EMP, what?

With new modeling and solution concepts do not:

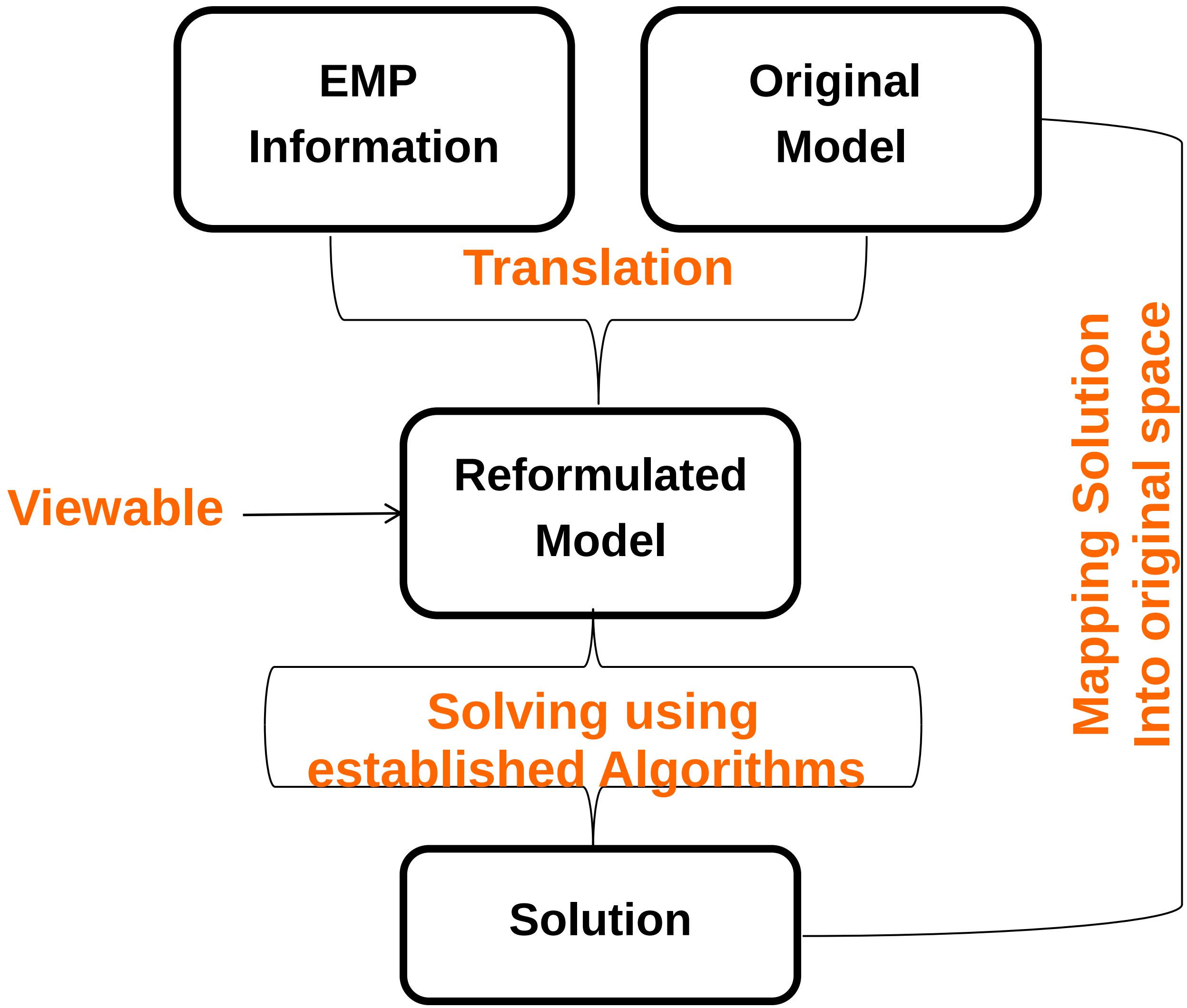
- overload existing GAMS notation right away !
- attempt to build new solvers right away !

But:

- Use existing language features to specify additional model features, structure, and semantics
- Express extended model in symbolic (source) form and apply existing modeling/solution technology
- Package new tools with the production system

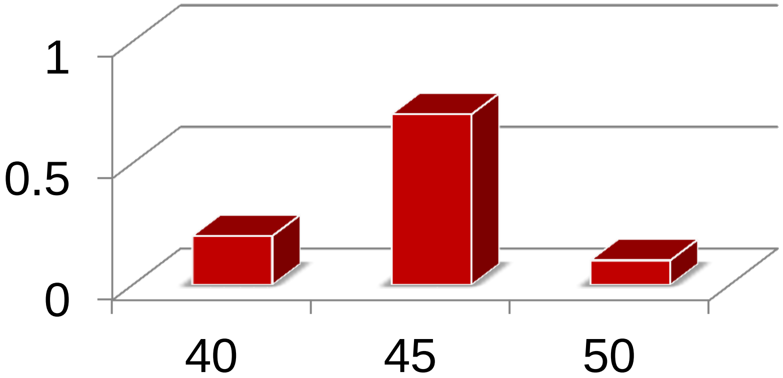
→ **Extended Mathematical Programming (EMP)**

JAMS: a GAMS EMP Solver

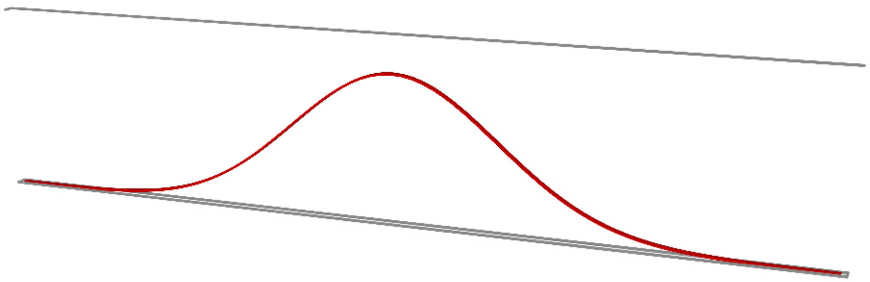


Random Variables

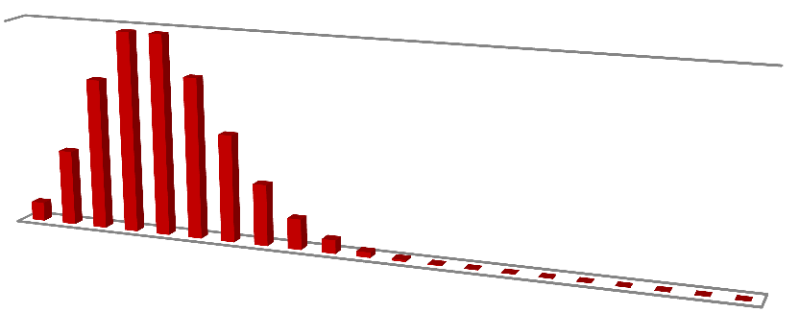
Discrete Distribution



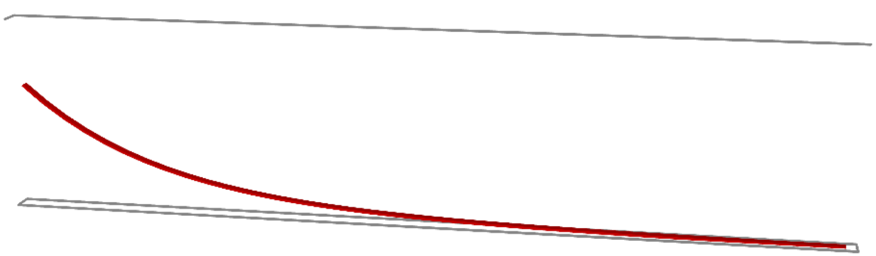
Normal Distribution



Poisson Distribution



Exponential Distribution



Random Variables (RV) [`randVar`]

- Defines both discrete and parametric random variables:

```
randVar rv discrete prob val {prob val}
```

- The distribution of discrete random variables is defined by pairs of the probability `prob` of an outcome and the corresponding realization `val`

```
randVar rv distr par {par}
```

- The name of the parametric distribution is defined by `distr`, `par` defines a parameter of the distribution

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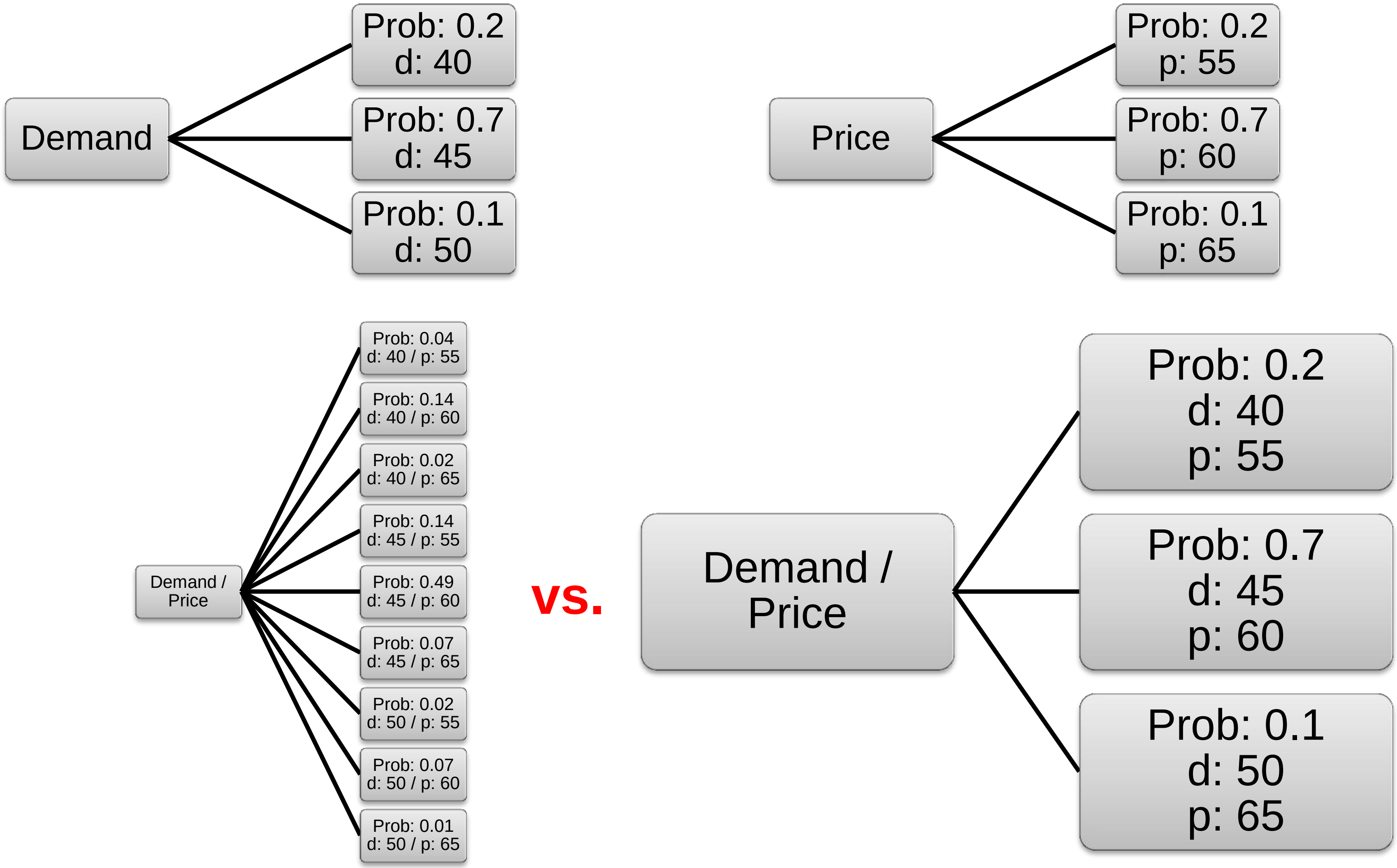
Joint Random Variables



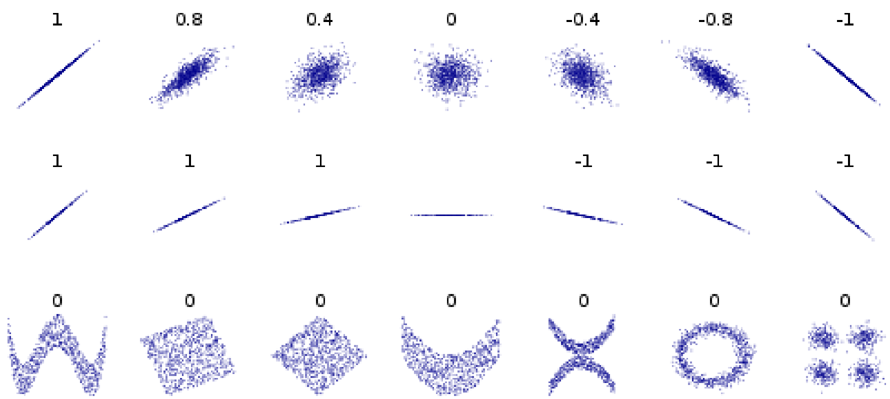
Joint RVs [jRandVar]

- Defines discrete random variables and their joint distribution:

```
jRandVar rv rv {rv} prob val val {val}
                   {prob val val {val}}
```

- At least two discrete random variables `rv` are defined and the outcome of those is coupled
- The probability of the outcomes is defined by `prob` and the corresponding realization for each random variable by `val`

Correlation between RVs [`correlation`]



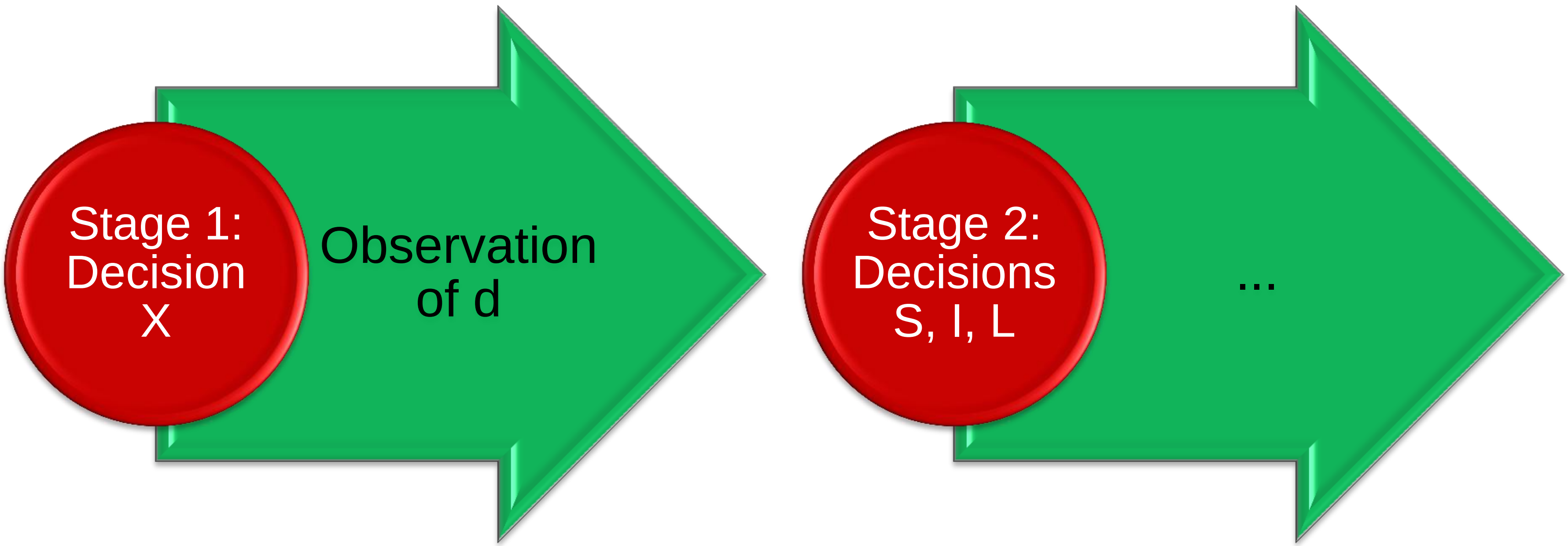
Source: Wikipedia

- Defines a correlation between a pair of random variables:

```
correlation rv rv val
```

- `rv` is a random variable which needs to be specified using the `randvar` keyword and `val` defines the desired correlation (-1 ≤ `val` ≤ 1)

Stages



Stages [`stage`]

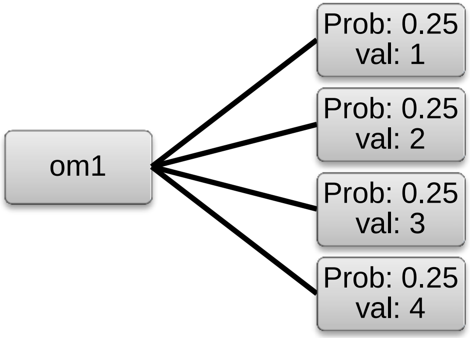
- Defines the stage of random variables (`rv`), equations (`equ`) and variables (`var`):

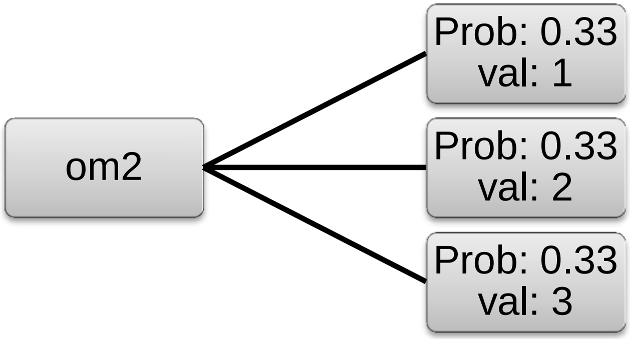
```
stage stageNo rv | equ | var {rv | equ | var}
```

- `StageNo` defines the stage number
- The default `StageNo` for the objective variable and objective equation is the highest stage mentioned
- The default `StageNo` for all the other random variables, equations and variables not mentioned is 1

Chance Constraints

```
OBJ.. Z =e= X1 + X2;
E1..  om1*X1 +   X2 =g=  7;
E2..  om2*X1 + 3*X2 =g= 12;
Model sc / all /;
solve sc min z use lp;
```






```
chance E1 0.6
chance E2 0.6
```

Chance Constraints

3 out of 4
must be true

[0.75 ≥ 0.6]

- 1*X1 + X2 =g= 7;
- 2*X1 + X2 =g= 7;
- 3*X1 + X2 =g= 7;
- 4*X1 + X2 =g= 7;

2 out of 3
must be true

[0.66 ≥ 0.6]

- 1*X1 + 3*X2 =g= 12;
- 2*X1 + 3*X2 =g= 12;
- 3*X1 + 3*X2 =g= 12;

Just in case: X1 = 2 and X2 = 3 are optimal.

Chance Constraints [`chance`]

- Defines individual or joint chance constraints (CC):

```
chance equ {equ} [holds] minRatio [weight|varName]
```

- Individual CC: A single constraint `equ` has to hold for a certain ratio (0 ≤ `minRatio` ≤ 1) of the possible outcomes
- Joint CC: A set of constraints `equ` has to hold for a certain ratio (0 ≤ `minRatio` ≤ 1) of the possible outcomes
- If `weight` is defined, the violation of a CC gets penalized in the objective (weight violationRatio)
- If `varName` is defined the violation get multiplied by this existing variable

Expected Value [**`ExpectedValue`**]

- This is the default objective:

```
ExpectedValue [x EV_x]
```

- If only `ExpectedValue` is defined, the expect value of the GAMS objective variable will be optimized (same as if it would be omitted at all)

- If the variable pair `x EV_x` is defined, GAMS will replace its objective variable by `EV_x`, which will become the expected value of `x`

Conditional Value at Risk [cVaR]

- As an alternative to the expected value, the conditional value at risk (cVaR) can be optimized:

```
cVaR [x cVaR_x] theta
```

- If only `cVaR theta` is defined, the cVaR of the GAMS objective variable to the quantile level `theta` will be optimized

- If the variable pair `x cVaR_x` is defined, GAMS will replace its objective variable by `cVaR_x`, which will become the cVaR of `x` to the quantile level `theta`

Combining EV and cVaR

It is also possible optimize a combination of the expected value and the conditional value at risk like this:

```
...
defobj..
  obj =e= lambda*EV_r + (1-lambda)*CVaR_r;

ExpectedValue r EV_r
cvarlo        r CVaR_r 0.1
...
```

Output Extraction

- The expected value of the solution can be accessed via the regular `.L` and `.M` fields
- In addition, the following information can be stored in a parameter by scenario:
  - `level`: Levels of variables or equations
  - `marginal`: Marginals of variables or equations
  - `randvar`: Realization of a random variable
  - `opt`: Probability of each scenario
- This needs to be stored in a separate dictionary:

```
Set dict / scen.scenario .''
           x   .level    .s_x
           ''  .opt      .srep /;
```

Adding Uncertainty to Transport

```
...
Model transport /all/ ;

file emp / '%emp.info%' /; put emp '* problem %gams.i%'/;
$onput
randvar b('new-york') normal 325 50
randvar b('chicago')  normal 300 50
randvar b('topeka')   normal 275 50
stage 2 b demand
$offput
putclose emp;

Set scen        scenarios / s1*s6 /;
Parameter
    s_b(scen,j)   demand realization by scenario
    s_x(scen,i,j) shipment per scenario
    s_s(scen) ;

Set dict / scen .scenario.''
           b    .randvar .s_b
           x    .level   .s_x /;

Solve transport using emp minimizing z scenario dict;
```

Summary

Available GAMS SP Solvers

	DE	DECIS	LINDO
chance	√		√
correlation			√
cVaR	√		
expectedValue	√		
jrandVar	√	√	√
randVar (discrete)	√	√	√
randVar (parametric)			√

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Conclusion

- Deterministic examples from all kind of application areas exist already (e.g. ~400 in the GAMS Model Library)

- Easy to add uncertainty to existing deterministic models, to
  - … either use specialized algorithms (DECIS, LINDO)
  - … or create Deterministic Equivalent and select from wide range of existing GAMS solver links (DE, free)

- New SP examples in the GAMS EMP Library

- More work to be done:
  - Scenario tree support
  - Sampling
  - …

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