Scenario tree generation for stochastic programming models using GAMS/SCENRED

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Overview

What is GAMS/SCENRED about

- GAMS/SCENRED is a link between the well-known General Algebraic Modeling System (GAMS) and the software tool SCENRED
- SCENRED provides a collection of software routines dealing with recent scenario tree manipulation algorithms in stochastic programming
- It is developed at the Department of Mathematics at Humboldt-University Berlin by the research group of Prof. Werner Römisch
- A first version of GAMS/SCENRED has been available since 2002
- Now we offer a basically extended version SCENRED2

What is new in SCENRED2

- SCENRED has been extended by scenario tree construction tools
- Available scenario reduction methods are improved by new metrics
- A lot of visualization functions (connected to GNUPLOT) are integrated

About Scenario Reduction

Introduction

- Stochastic programs deal with finite sets of scenarios to model the probabilistic information on random data
- The number of scenarios could be very large
- Scenario reduction becomes important to reduce the high sized scenario based models to make them numerical tractable

$\Rightarrow$ Scenario reduction aims to reduce the number of scenarios and to maintain the probability information as good as possible!

About Scenario Reduction

Probability metrics

- To control the probability information probability metrics are needed
- Optimal values behave stable with respect to small perturbations of the underlying distribution in terms of probability metrics of the form

$$d_{\mathcal{F}}(P,Q) = \sup_{f\in\mathcal{F}} \left| \int_{\Xi} f(\xi)P(d\xi) - \int_{\Xi} f(\xi)Q(d\xi) \right|$$

Linear case:

$$\mathcal{F}_c := \left\{ f : \Xi \to \mathbb{R} \,\middle|\, f(\xi) - f(\tilde{\xi}) \leq c(\xi,\tilde{\xi}) \text{ for all } \xi,\tilde{\xi} \in \Xi \right\}$$

$$c(\xi,\tilde{\xi}) := \max\left\{ 1, \|\xi - \xi_0\|^{r-1}, \|\tilde{\xi} - \xi_0\|^{r-1} \right\} \|\xi - \tilde{\xi}\|$$

Metrics of this type are called *Fortet-Mourier metrics* of order r

About Scenario Reduction

Dual representation

- The dual representation of probability metrics are of the form

$$\mu_c(P,Q) \;=\; \inf \left\{ \int_{\Xi\times\Xi} c(\xi,\tilde{\xi})\eta(d\xi,d\tilde{\xi}) \;:\; \eta \in M(P,Q) \right\}$$

  These are the *Monge-Kantorovich transport functionals*

- It holds

$$d_{\mathcal{F}_c}(P,Q) \le \mu_c(P,Q) \quad \text{and} \quad d_{\mathcal{F}_c}(P,Q) = \mu_{\hat{c}}(P,Q)$$

  for the so-called *reduced costs* $\hat{c}$ with

$$\hat{c}(\xi,\tilde{\xi}) := \inf \left\{ \sum_{j=1}^{n+1} c(z_{j-1},z_j) : z_0 = \xi,\; z_{n+1} = \tilde{\xi},\; z_j \in \Xi,\; n \in \mathbb{N} \right\}$$

⇒ SCENRED2 allows to control the scenario reduction w.r.t. both the Fortet-Mourier metric and the Monge-Kantorovich functional!

About Scenario Reduction

- The dual reformulation allows to compute the scenario reduction without solving the underlying transport problem
- The problem of optimal scenario reduction is to find convenient scenarios for removing and it can be stated as

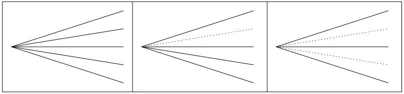
$$\min\left\{ D_J := \sum_{i\in J} p_i \min_{j\notin J} \hat{c}(\xi^i,\xi^j) \;\middle|\; J\subset\{1,\dots,N\},\ \#J = N-n \right\}$$

Approximative solutions by fast (heuristic) algorithms

Backward Reduction:

Delete scenario u^k such that

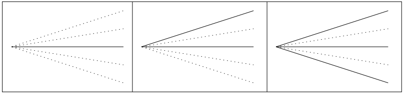
$$D_{J^{k-1}\cup\{u^k\}} = \min_{u\notin J^{k-1}} D_{J^{k-1}\cup\{u\}}$$



Forward Selection:

Select scenarien u^k such that

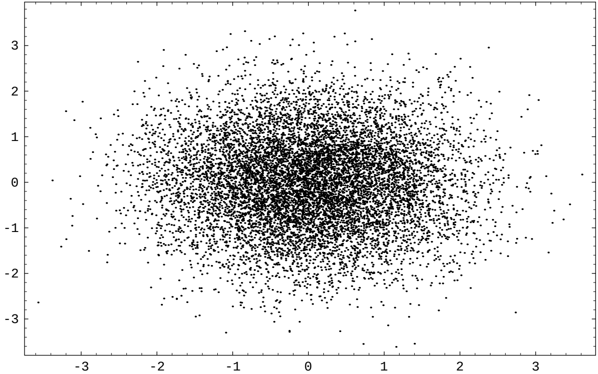
$$D_{J^{k-1}\setminus\{u^k\}} = \min_{u\in J^{k-1}} D_{J^{k-1}\setminus\{u\}}$$



About Scenario Reduction

Example 2-dimensional normal distribution

- Scenario reduction of the normal distribution from 10 000 scenarios to 20



About Scenario Reduction

Example 2-dimensional normal distribution

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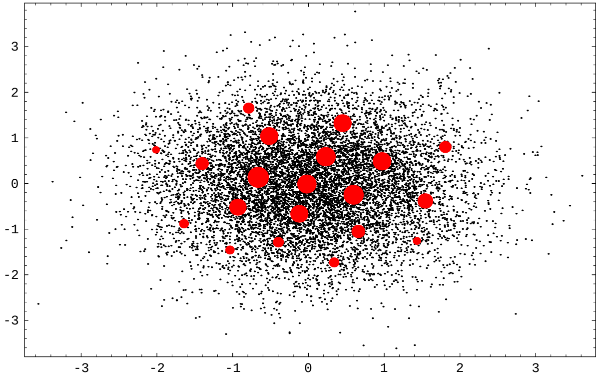
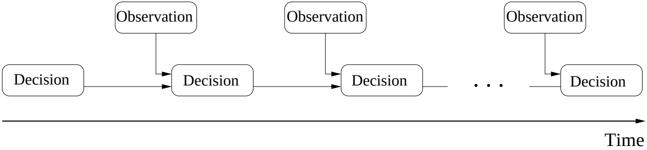


Scenario Tree Construction

Introduction

- We concider a multiperiod decision problem of the form



- We have a time discrete stochastic input process $\xi = (\xi_1, \ldots, \xi_T)$
- We introduce a decision process $x = (x_1, \ldots, x_T)$, where the stage decision x_t only depends on outcomes $\xi_1, \ldots, \xi_t$

$\Rightarrow$ Observation: A multistage stochastic program implies a certain information structure: $\mathcal{F}_1(\xi) \subseteq \ldots \subseteq \mathcal{F}_t(\xi) \subseteq \ldots \subseteq \mathcal{F}_T(\xi)$

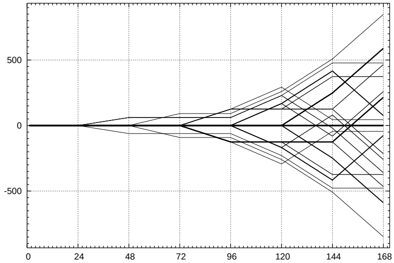
($\mathcal{F}_t(\xi)$ denotes the σ-field generated by $(\xi_1, \ldots, \xi_t)$)

Scenario Tree Construction

Assuming that the support of ξ is infinite:

- We have an infinite dimensional optimization problem
- Optimization problem is intractable in general

Replace ξ by a scenario tree approximation ξ_{tr} (finite distribution)



How does the optimal value change when ξ is replaced by ξ_{tr} ?

Scenario Tree Construction

Theorem (Stability – Heitsch/Römisch/Strugarek 06)

Under some regularity assumptions it holds for $\|\xi - \tilde{\xi}\|_r < \delta$:

$$|v(\xi) - v(\tilde{\xi})| \leq L \left(\|\xi - \tilde{\xi}\|_r + D_{\mathrm{f}}(\xi, \tilde{\xi}) \right),$$

where $v(\cdot)$ denotes the optimal value and $D_{\mathrm{f}}(\xi, \tilde{\xi})$ is a distance of the filtrations defined by ξ and $\tilde{\xi}$, respectively.

The filtration (information) distance is defined by

$$D_{\mathrm{f}}(\xi, \tilde{\xi}) := \inf_{\substack{x \in S(\xi) \\ \tilde{x} \in S(\tilde{\xi})}} \sum_{t=2}^{T-1} \max \left\{ \|x_t - \mathbb{E}[x_t | \mathcal{F}_t(\tilde{\xi})]\|_{r'}, \|\tilde{x}_t - \mathbb{E}[\tilde{x}_t | \mathcal{F}_t(\xi)]\|_{r'} \right\}$$

Here $S(\xi)$ denotes the solution set of the model with input ξ.

Scenario Tree Construction

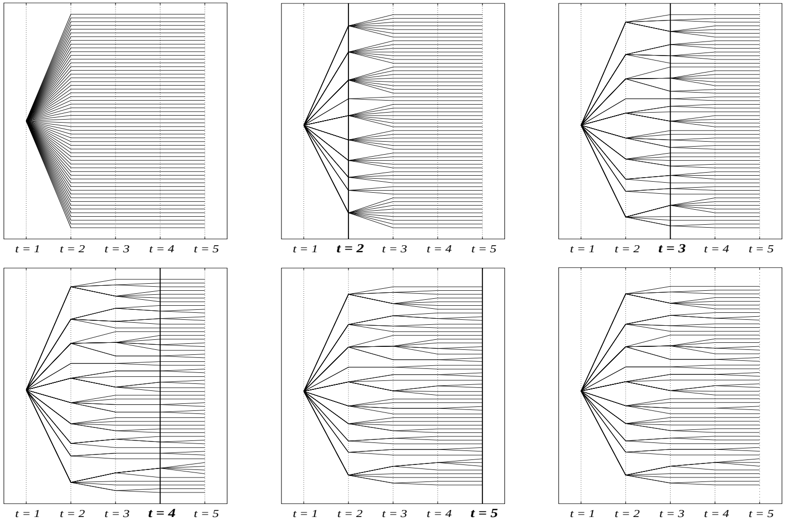
General approach

1. Providing of scenarios ξ^i with probabilities p_i, $i = 1, \ldots, N$:
   - Adaption of a statistic model for the underlying data process (decomposition of historical data, cluster analysis, time series models, stress scenarios)
   - Simulation of scenarios out of the statistic model (may be a large number of scenarios)

2. Construction of the scenario tree out of scenarios ξ^i based on *stagewise approximations*:
   - Choose a construction ε-percentage (should depend on the number of scenarios)
   - Determine a scenario tree ξ_{tr} by recursive scenario reduction (both the probability distance and the filtration distance can be controlled by this approach)

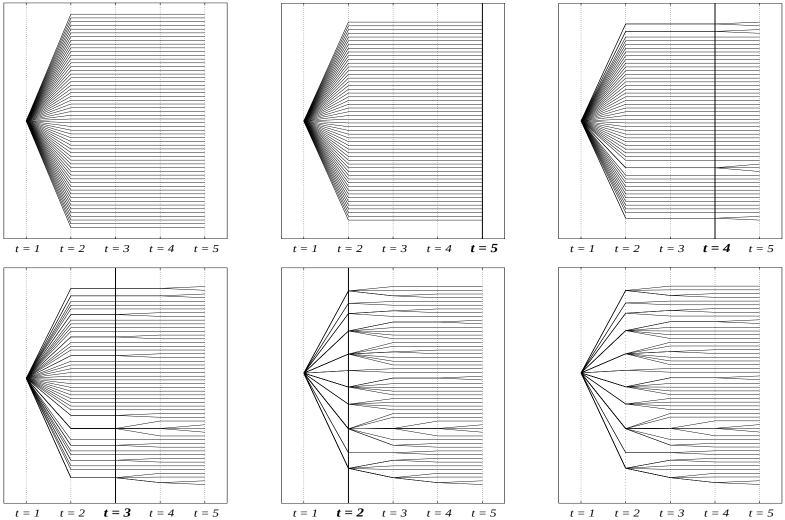
Scenario Tree Construction

Recursive forward scenario reduction:



Scenario Tree Construction

Recursive backward scenario reduction:



Example Problem

Stochastic purchase problem

$$\min \left\{ \mathbb{E}\left[\sum_{t=1}^{3} \xi_t x_t\right] \;\middle|\; \begin{array}{l} x_t \geq 0, \\ x_t \text{ is } \mathcal{F}_t(\xi)\text{-measurable}, \\ x_1 + x_2 + x_3 \geq 1 \end{array} \right\}$$

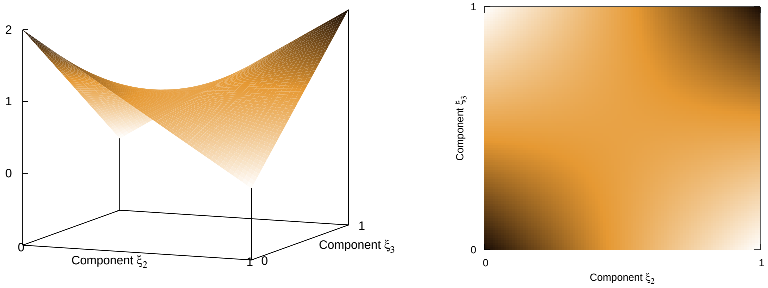
Assumptions

- ξ_1 is deterministic and $\xi_1 \equiv 1$
- $\xi_2 \sim U([0,1])$ (uniformly distributed)
- $\xi_3 \sim L([0,1])$ (linear distributed) with slope depending on ξ_2:

$$\mathbb{P}\left(\xi_3 \in [a,b] \,\middle|\, \xi_2 = x\right) = \int_a^b \left[2(1-x) - 2(1-2x)y\right] dy$$

Example Problem

Probability distribution



Joint density function of the stochastic components (ξ_2, ξ_3)

Example Problem

Analytical solution

Optimal decision

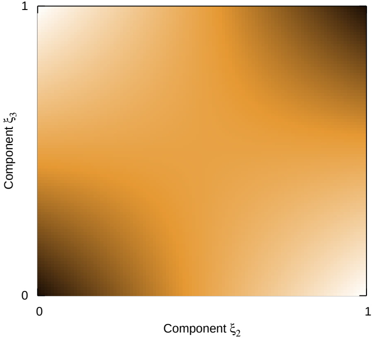
$$x_1 \equiv 0, \qquad x_2 = \begin{cases} 1 & , \text{ if } \xi_2 \leq \frac{1}{2} \\ 0 & , \text{ otherwise} \end{cases} , \qquad x_3 = \begin{cases} 1 & , \text{ if } \xi_2 > \frac{1}{2} \\ 0 & , \text{ otherwise} \end{cases}$$

Optimal value (OPT) / Value of perfect information (VOPI)

$\text{OPT} = 0.4167$ $\qquad$ $\text{VOPI} = 0.3667$

Example Problem

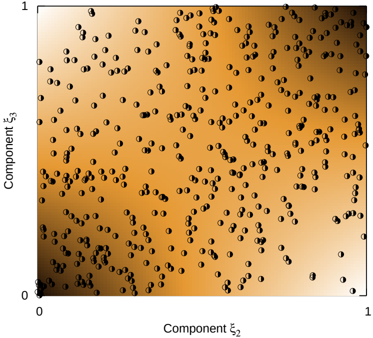
Construction of the scenario tree

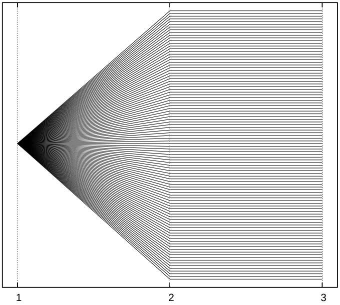


Example Problem

Construction of the scenario tree

- Simulation of a scenario sample $\xi^1, \ldots, \xi^N$ based on the distribution

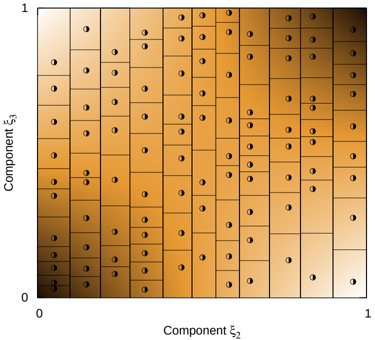


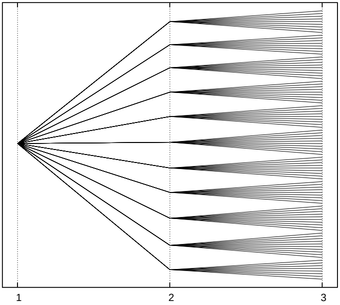


Example Problem

Construction of the scenario tree

- Simulation of a scenario sample $\xi^1, \ldots, \xi^N$ based on the distribution
- GAMS/SCENRED2 allows to generate a scenario tree





Example Problem

$OPT^* = 0.4167$ / $VOPI^* = 0.3667$

Numerical results

Sample	Scenarios	Tree size	OPT	VOPI
A	500	200	0.4156	0.3629
		100	0.4179	
		50	0.4164	
B	500	200	0.4162	0.3640
		100	0.4197	
		50	0.4179	
C	500	200	0.4245	0.3749
		100	0.4261	
		50	0.4246	
D	500	200	0.4092	0.3632
		100	0.4121	
		50	0.4114	

Using GAMS/SCENRED

Organization of the GAMS program

1. Data:
   - set & parameter declarations and definitions
   - include SCENRED symbols by: $libinclude scenred.gms
   - setup scenarios (by nodes) and options for SCENRED run
2. SCENRED call:
   - export data from GAMS to SCENRED (using GDX unload)
   - execute SCENRED or SCENRED2
   - import data from SCENRED to GAMS (using GDX load)
3. Model:
   - variable & equation declarations and definitions
   - model definitions using node subsets of reduced/constructed tree
   - solve the model

Example: An implementation of the stochastic *purchase example problem* is available as GAMS program ('*srpurchase.gms*')