Approximation of Non-linear Functions in Mixed Integer Programming

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Joint work with Markus Möller and Susanne Moritz

Outline

1. Non-linear Functions in MIPs
   - design of sheet metal
   - gas optimization
   - traffic flows
2. Modelling Non-linear Functions
   - with binary variables
   - with SOS constraints
3. Polyhedral Analysis
4. Computational Results

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Design of Transport Channels

Goal

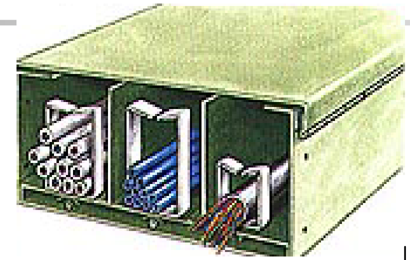
Maximize stiffness

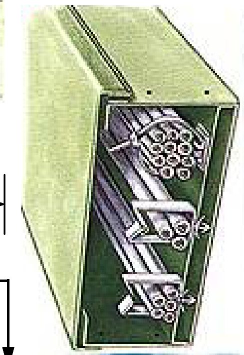
Subject To

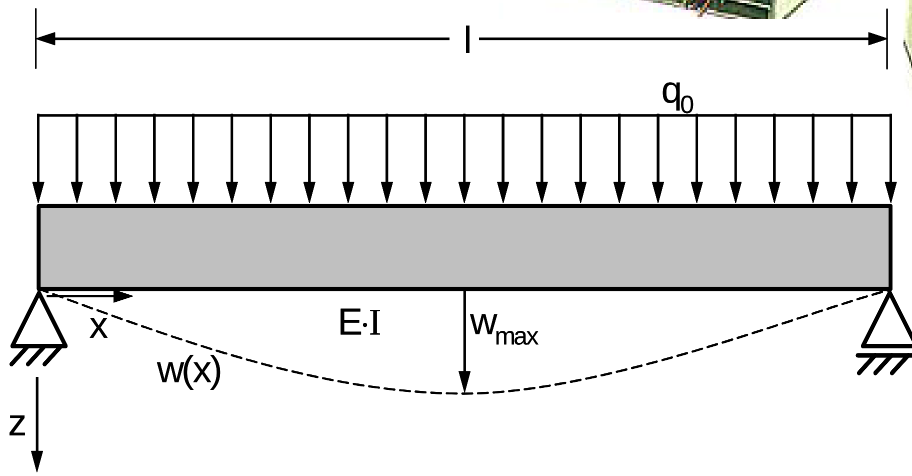
- Bounds on the perimeters
- Bounds on the area(s)
- Bounds on the centre of gravity

Variables

- topology
- material



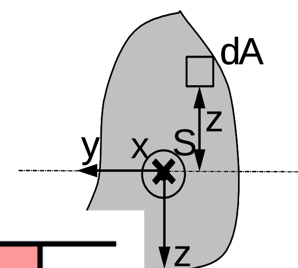


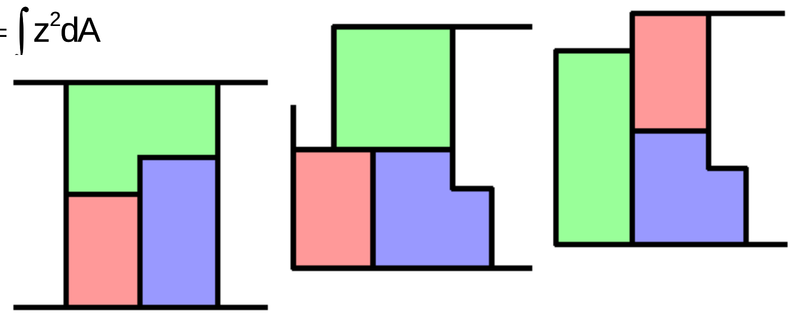


$$w(x) = \frac{q_0 \cdot l^4}{24 \cdot E \cdot I_y} \cdot \left[\left(\frac{x}{l}\right)^4 - 2 \cdot \left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right) \right]$$

$$w_{max} = w\left(\frac{l}{2}\right) = \frac{5}{384} \cdot \frac{q_0 \cdot l^4}{E \cdot I_y}$$

$$I_y = \int z^2 dA$$





Optimization of Gas Networks

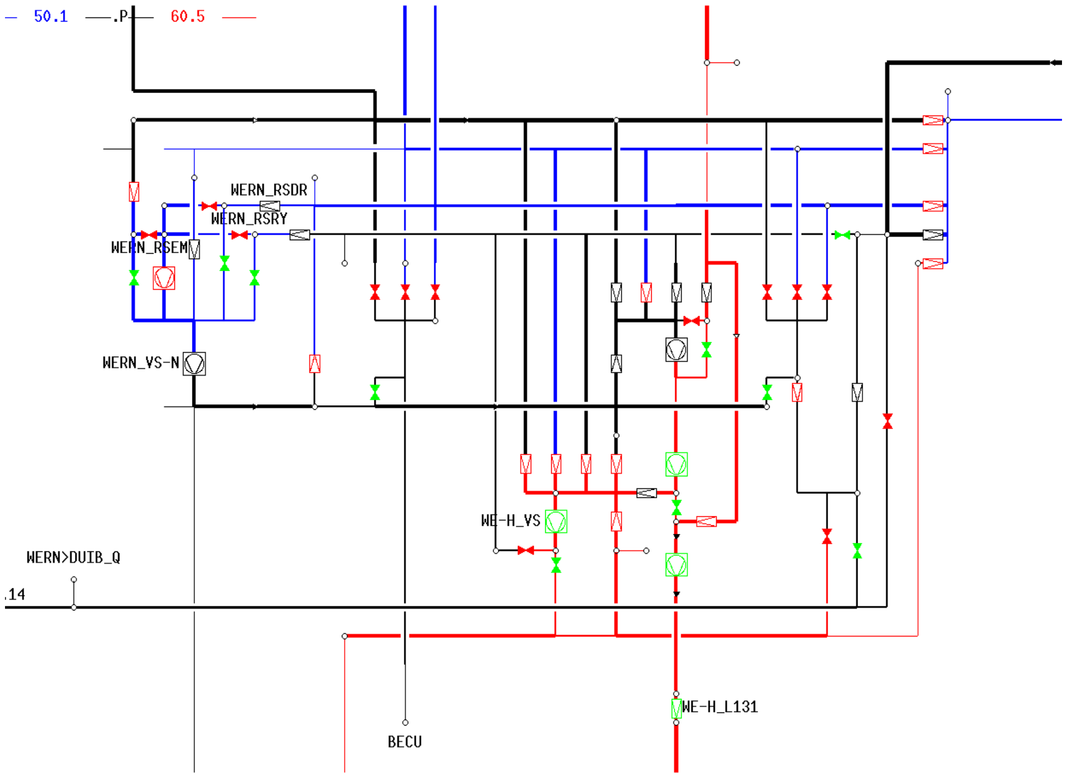
Goal

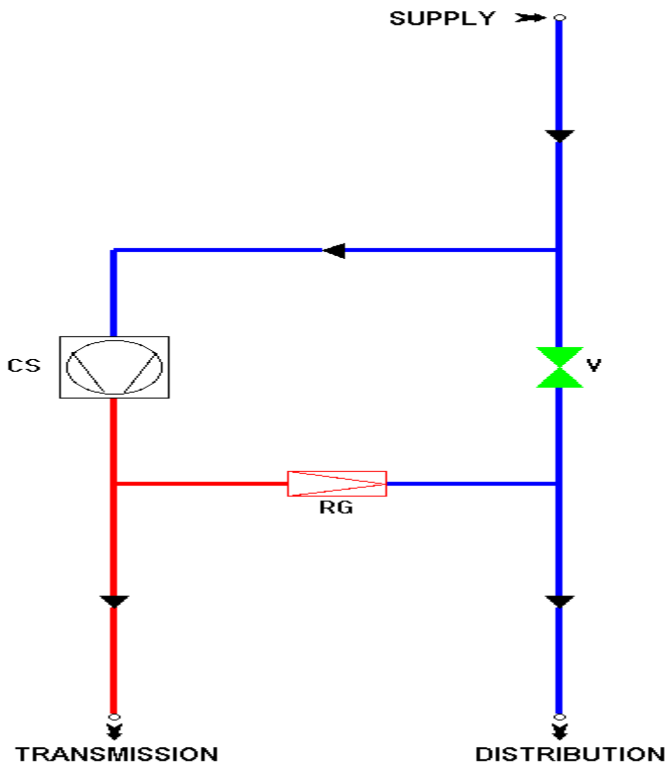
Minimize fuel gas consumption

Subject To

- contracts
- physical constraints

Gas Network in Detail



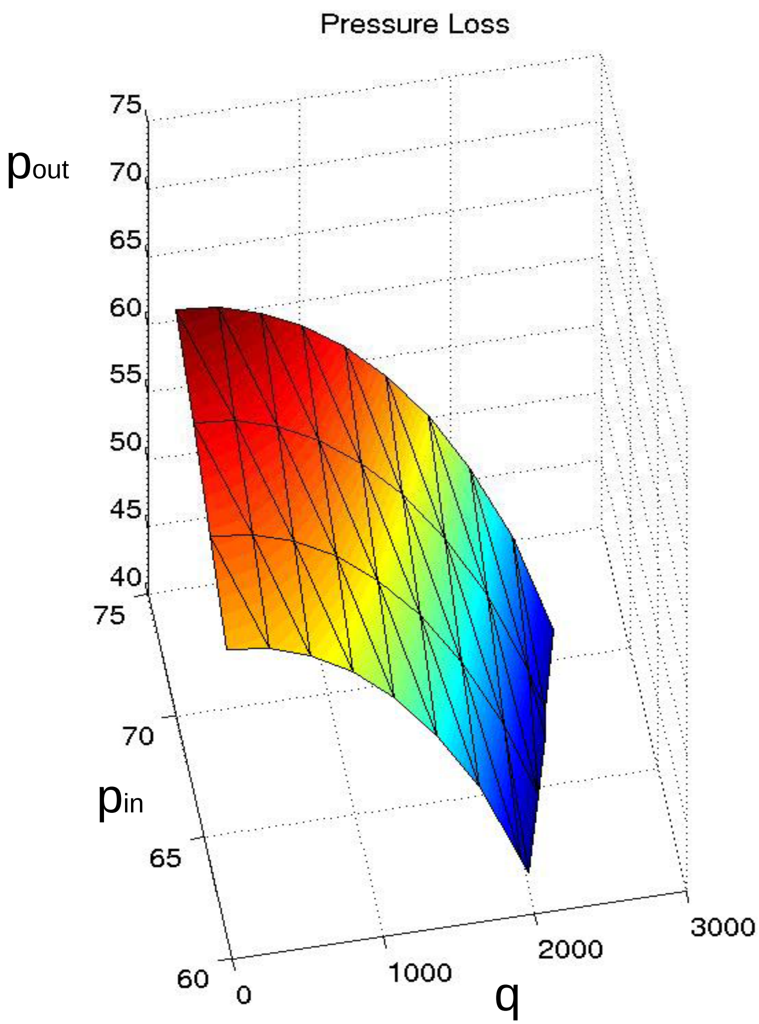


Gas Networks: Nature of the Problem

- Non-linear
  - fuel gas consumption of compressors
  - pipe hydraulics
  - blending, contracts

- Discrete
  - valves
  - status of compressors
  - contracts

Pressure Loss in Gas Networks



$$\partial_x p + \frac{\lambda \rho_o p_o T}{2 D A^2 z_o T_o} \frac{|q| q z}{p}$$

$$+ \frac{\rho_o p_o T}{A^2 z_o T_o} \partial_x \left(\frac{q^2 z}{p}\right) + \frac{\rho_o}{A} \partial_t q$$

$$+ \frac{g \rho_o z_o T_o}{p_o T} \partial_x \left(\frac{h p}{z}\right) = 0$$

horizontal pipes ⇓ stationary case

$$p_{out}^2 = p_{in}^2 - \text{ff}\ |q| q$$

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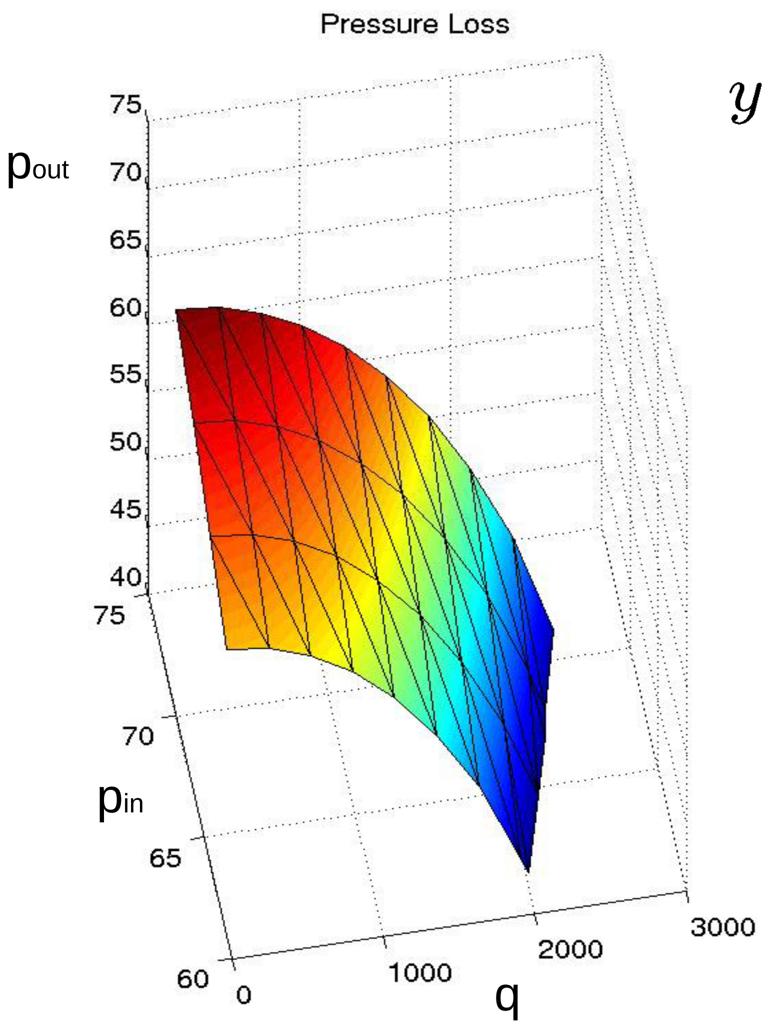
Approximation of Pressure Loss: Binary Approach



$$\lambda^i \geq 0, \quad \text{for grid point} \quad i \in \Lambda$$
$$y^i \in \{0,1\}, \quad \text{for triangle} \quad j \in Y$$

$$(1) \quad \sum_{i \in \Lambda} \lambda^i = 1$$

$$(2) \quad \sum_{j \in Y} y^j = 1$$

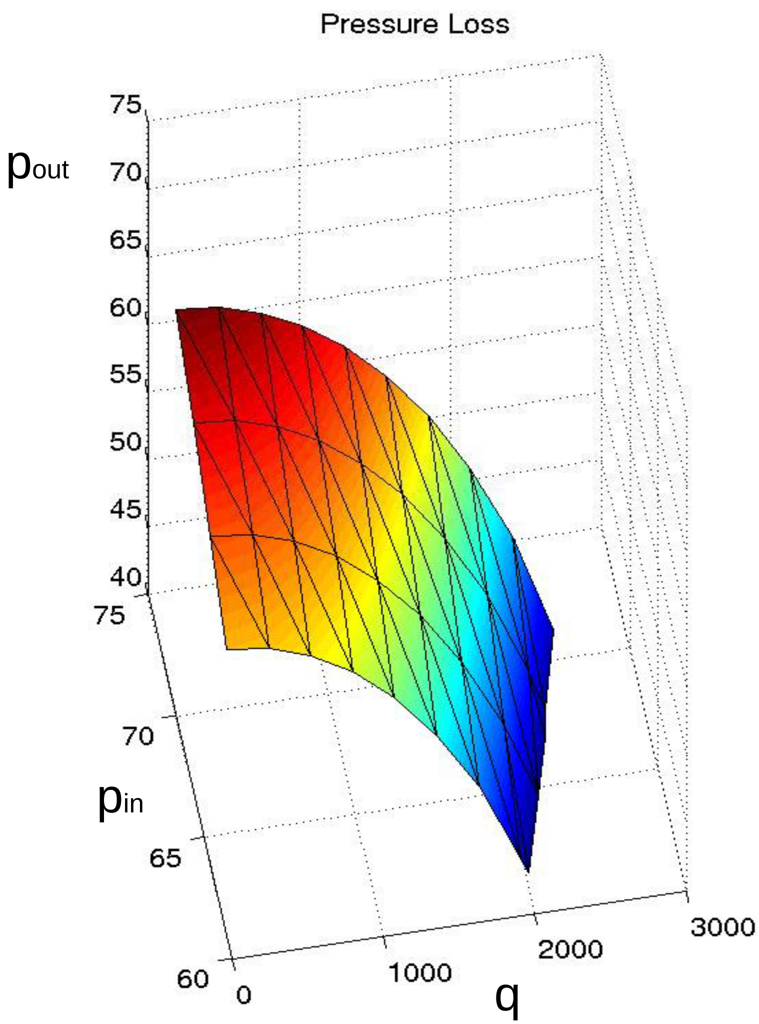
$$(3) \quad y^j \leq \sum_{k \in N^j} \lambda^k$$

$$(4) \quad p_{in} = \sum_{i \in \Lambda} p_{in}^i \lambda^i$$

$$(5) \quad p_{out} = \sum_{i \in \Lambda} p_{out}^i \lambda^i$$

$$(6) \quad q = \sum_{i \in \Lambda} q^i \lambda^i$$

Approximation of Pressure Loss: SOS Approach



$$p_{in} = \sum_{i \in \Lambda} p_{in}^{i} \lambda^{i}$$

$$q = \sum_{i \in \Lambda} q^{i} \lambda^{i}$$

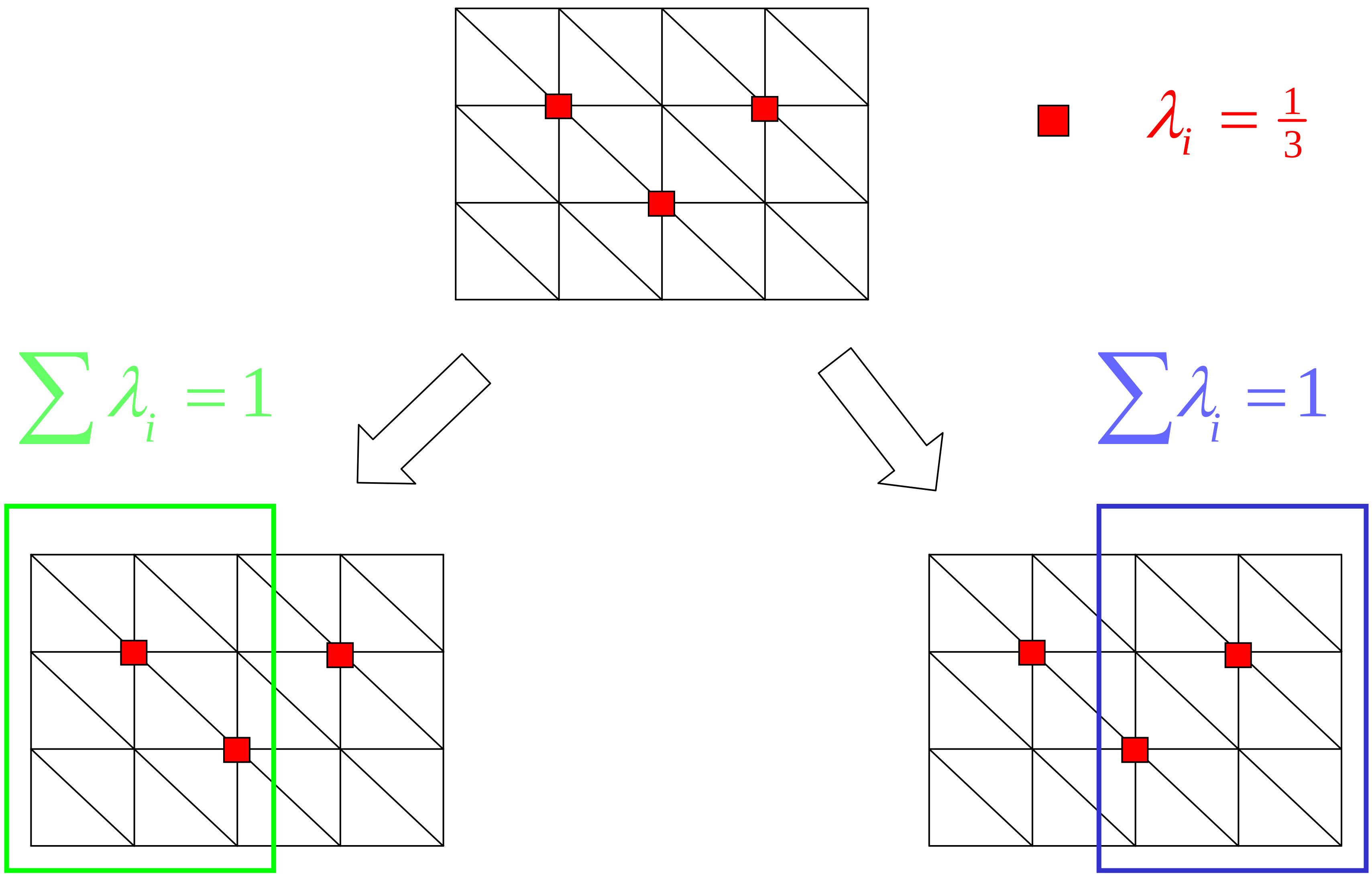
$$p_{out} = \sum_{i \in \Lambda} p_{out}^{i} \lambda^{i}$$

$$\sum_{i \in \Lambda} \lambda^{i} = 1 \quad (*)$$

$$\lambda^{i} \geq 0$$

$(*)$ must meet the triangle condition

Branching on SOS Constraints



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The SOS Constraints: General Definition

Given

- grid points $\Lambda = \{1, \ldots, n\}$ and
- a set of subsets $Y = \{N^1, \ldots, N^d\}$, $N^i \subset \Lambda$.

A vector λ satisfies the **set condition** for Y and

$$\sum_{i \in \Lambda} \lambda^i = 1 \qquad (*)$$

if $\{i \in \Lambda : \lambda^i > 0\} \subseteq N^r$ for some r.

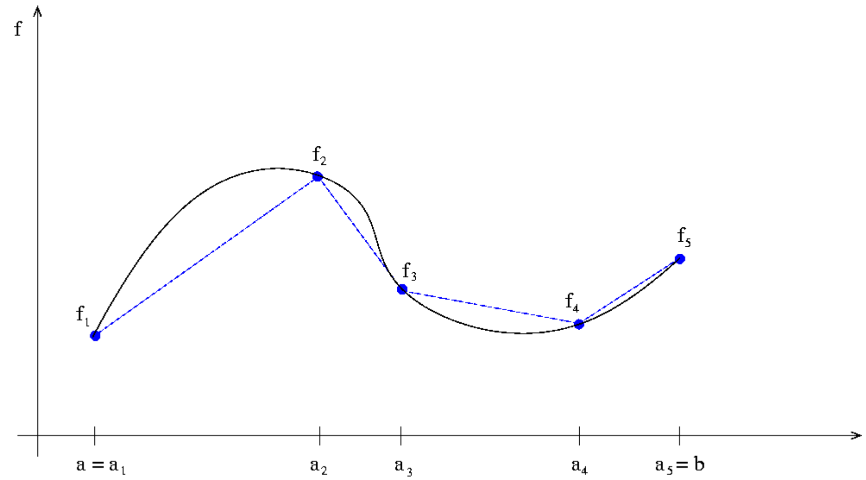
$(*)$ is called **SOS constraint of Type k**, where

$$k = \max_i |N^i|.$$

The SOS Constraints: Special Cases

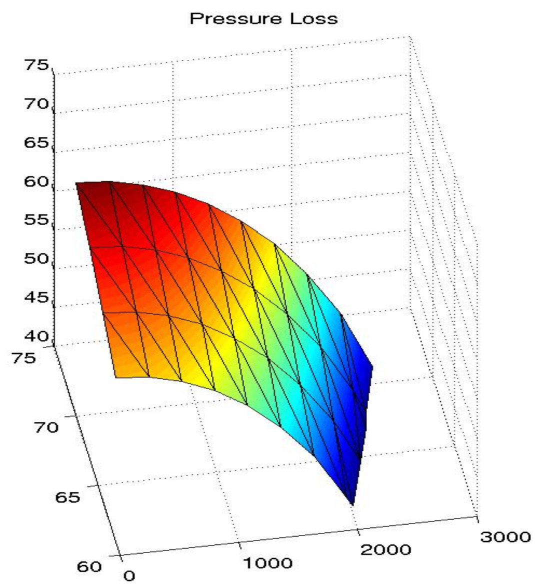
- SOS Type 2 constraints

$$N^i = \{i, i+1\}$$



- SOS Type 3 constraints

$$|N^i| = 3 \quad \forall i$$

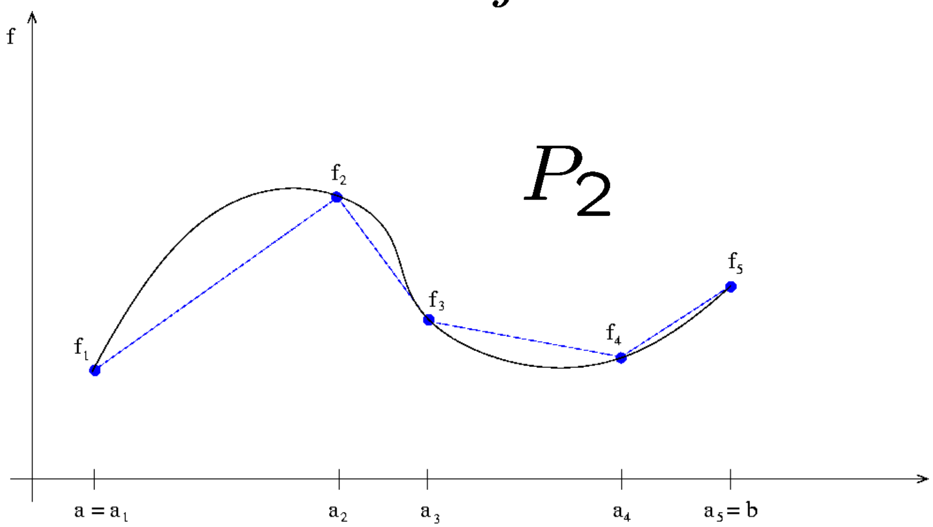


The Binary Polytope

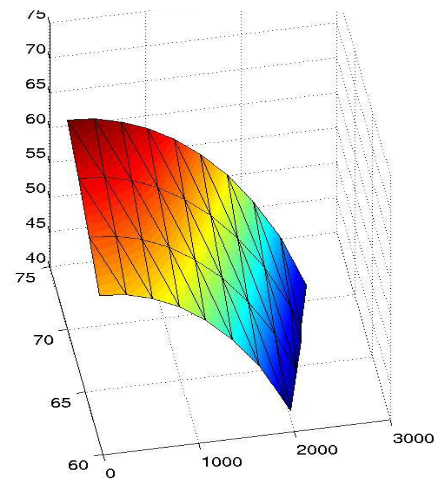
Let $P_k = \text{conv}\,\{\lambda \in \mathbb{R}^{\Lambda}_{+},\, y \in \{0,1\}^{Y} :$

$$\begin{array}{llcl} (1) & \sum_{i\in\Lambda} \lambda^i & = & 1 \\ (2) & \sum_{j\in Y} y^j & = & 1 \\ (3) & y^j & \leq & \sum_{i\in N^j} \lambda^i \,\} \end{array}$$

with $k = \max_j |N^j|$



P_2

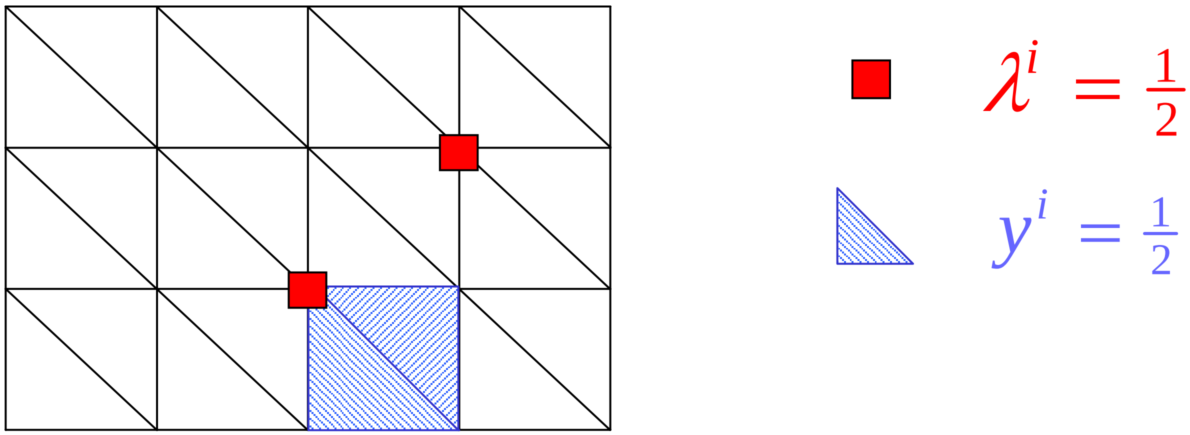


P_3

The Binary Polytope: Inequalities

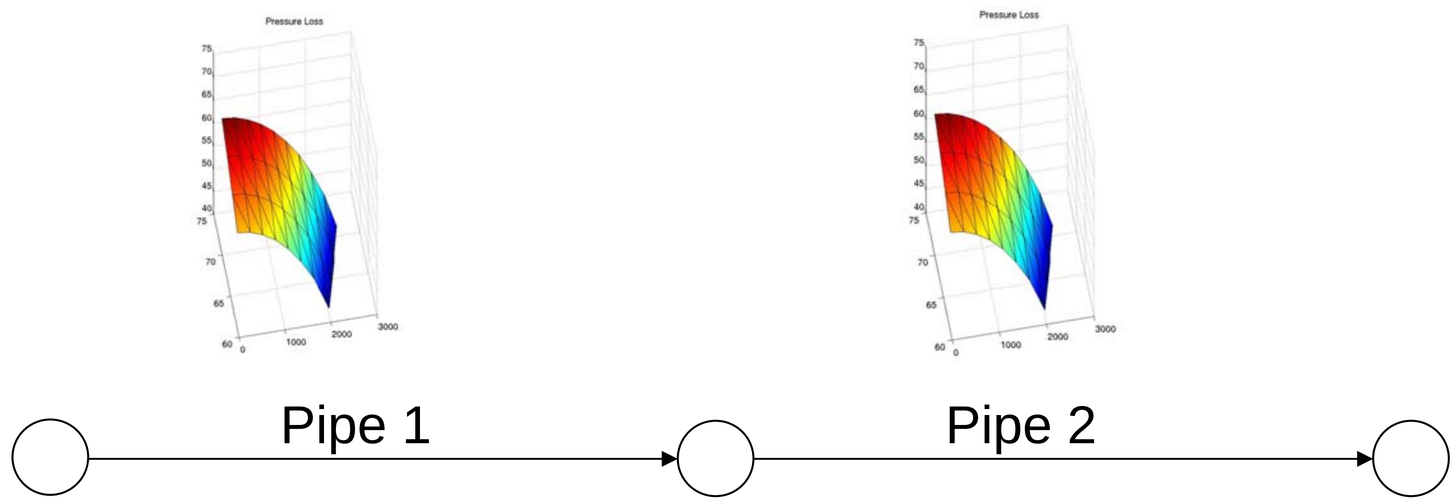
For $\emptyset \neq J \subset Y$ and $I := \bigcup_{j \in J} N^j$ let

$$\sum_{j \in J} y^j \leq \sum_{i \in I} \lambda^i \qquad (*)$$



Theorem. $(*)$ describes P_2 and P_3 completely.

The SOS Polytope



$$P_\Delta = \left\{ \binom{\lambda_1}{\lambda_2} \in \mathbb{R}^{|\Lambda_1|+|\Lambda_2|} \;\middle|\; \begin{array}{rcl} \sum\limits_{j\in\Lambda_1} \lambda_1^j & = & 1 \\ \sum\limits_{j\in\Lambda^2} \lambda_j^2 & = & 1 \\ \sum\limits_{j\in\Lambda_1} p_1^j \lambda_1^j - \sum\limits_{j\in\Lambda_2} p_2^j \lambda_2^j & = & 0 \\ \lambda_1^j, \quad \lambda_2^j & \geq & 0 \\ \lambda_1, \lambda_2 \text{ satisfy the set condition for } Y_1 \text{ and } Y_2 & & \end{array} \right\}.$$

The SOS Polytope: Increasing Complexity

$\|\Delta\|$	$\|Y\|$	Vertices	Facets	Max. Coeff.
8	12	16	18	25
16	18	49	47	42
24	24	73	90	670
32	32	142	10492	50640

The SOS Polytope: Properties

Theorem. There exist only polynomially many vertices $v_1, \ldots, v_l$ with $l \leq 9|Y_1|\,|Y_2|$.

- The vertices can be determined algorithmically
- This yields a polynomial separation algorithm by solving for given λ_1^* and λ_2^*

$$\max \quad a^T \binom{\lambda_1^*}{\lambda_2^*} - \alpha$$

$$\text{s.t.} \quad a^T v_i \leq \alpha \qquad \text{for } i = 1, \ldots, k$$

The SOS Polytope: Generalizations

- Pipe to pipe with respect to pressure and flow
- Several pipes to several pipes
- Pipes to compressors (SOS constraints of Type 4)
- General Mixed Integer Programs:

Consider *Ax=b* and a set I of SOS constraints of Type k_i for $i \in I$ such that each variable is contained in exactly one SOS constraint. If the rank of *A* (incl. I) and $\max_i k_i$ are fixed then

$$P = \text{conv}\left\{x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} Ax = b \\ x \text{ satisfies the set} \\ \text{condition for } i \in I \end{array}\right\}$$

has only polynomial many vertices.

Binary versus SOS Approach

- Binary
  - more (binary) variables
  - more constraints
  - complex facets
  - LP solutions with fractional *y* variables and correct λ variables
- SOS
  - \+ no binary variables
  - \+ triangle condition can be incorporated within branch & bound
  - \+ underlying polyhedra are tractable

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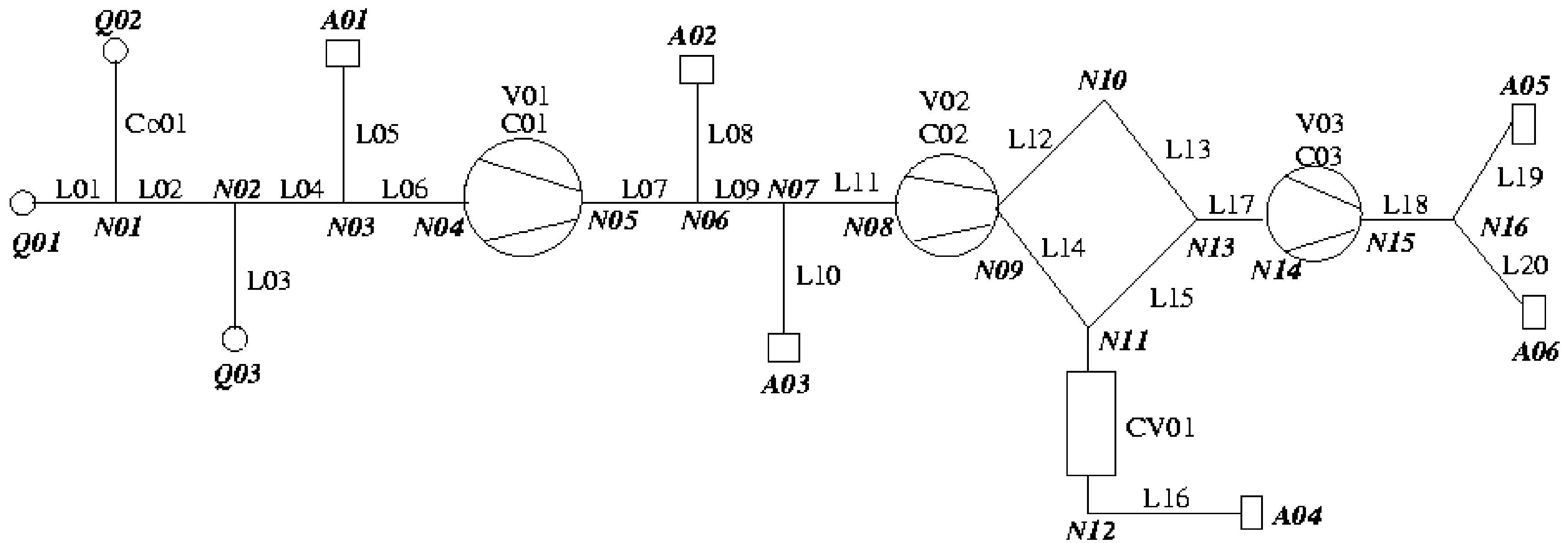

Nr of Pipes	Nr of Compressors	Total length of pipes	Time (ε = 0.05)	Time (ε = 0.01)
11	3	920	1.2 sec	2.0 sec
20	3	1200	1.2 sec	9.9 sec
31	15	2200	11.5 sec	104.4 sec