Mixed Complementarity Formulations of Stochastic Equilibrium Models with Recourse

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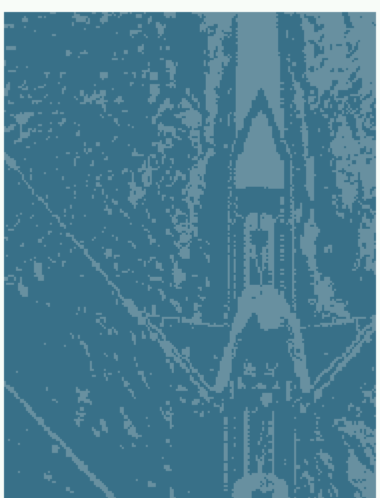

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MASTER 8+ MEN

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2 Penn AC
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4 Palm Beach Rowing Assn.
5 Fat Cat Rowing Club
6 Vesper Boat Club
7 Union Boat Club
8 Bulldog Rowing Club
9 Brigantine Rowing Club
10 RIC Amsterdam
11 Warren Rowing Club
12 Four Score & Four RC
13 X-Press Boat Club
14 Wide Load Boat Club
15 Georgian Bay RC
16 Free Radical Rowing Club
17 Purple Bull Roklub
18 Gentle Giant Rowing
19 Cork Boat Club
20 Water Street Rowing
21 River Rowing Association
22 Curlew Rowing Club
23 Three Rivers Rowing Assn.
24 Ann Arbor Rowing Club
25 Navy Masters Rowing Club

Three illustrative models

- Lifecyle consumption-savings decisions with income uncertainty (finite horizon NLP)
- Ramsey growth model with uncertain technology change (infinite-horizon NLP)
- Growth model with anticipated tax policy and parameter uncertainty (infinite-horizon MCP)

Programming Techniques

- Event tree management logic tool (probtree)
- Tight formulations
- Graphical tools for debugging (treeplot) and reporting (fanplot)
- Complementarity programming in a stochastic framework

Mixed Complementarity Problem - MCP Definition

Given: $F : R^n \rightarrow R^n \ , \quad \ell, u \in R^n$

Find: $z, w, v \in R^n$

such that
$$F(z) = w - v$$
$$\ell \leq z \leq u, \quad w \geq 0, \quad v \geq 0$$
$$w^T(z - \ell) = 0 \ , \quad v^T(u - z) = 0.$$

Credits

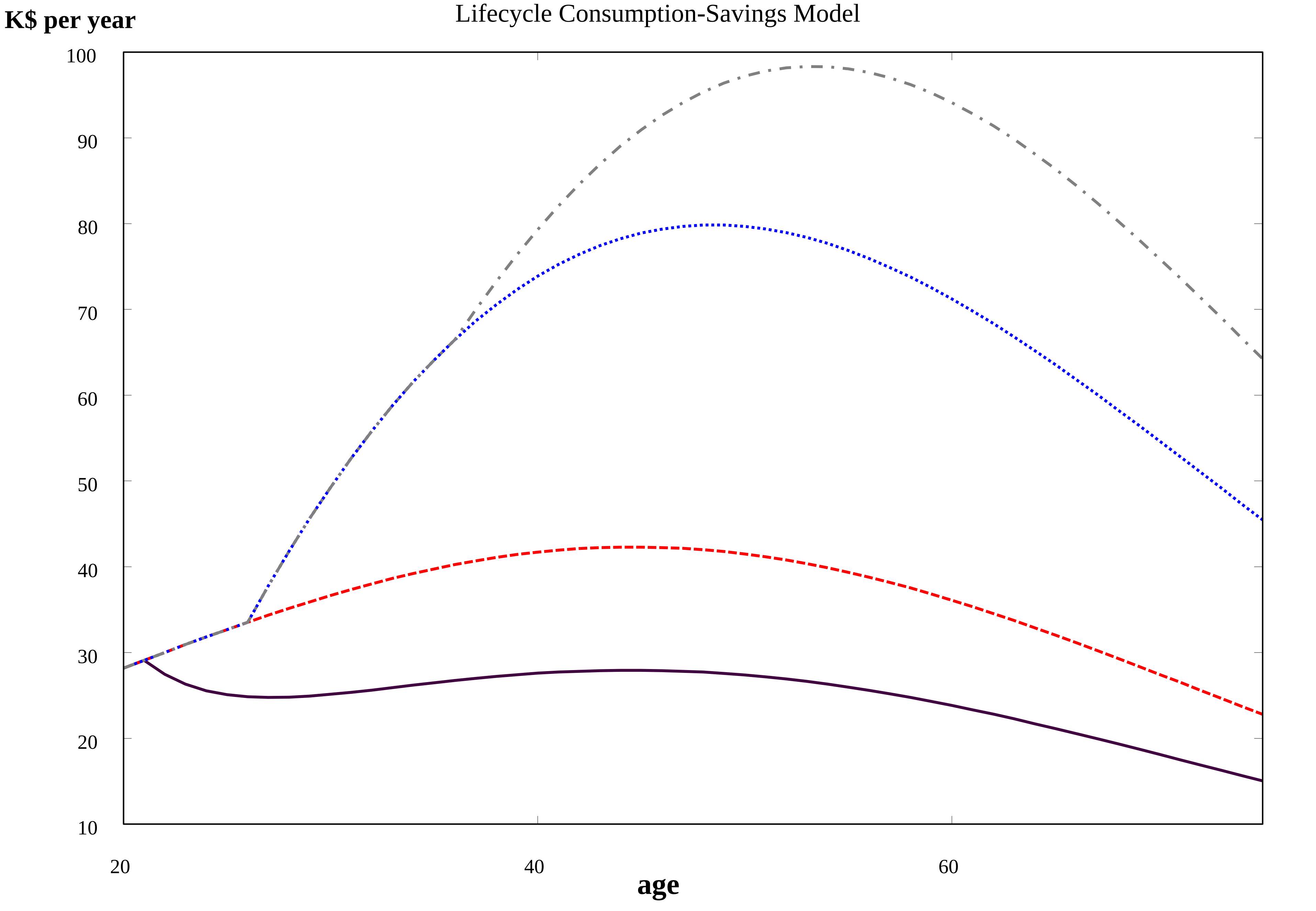
- DMR School
  - Sherman Robinson, Jaime deMelo,Kemal Dervis
- MPSGE School
  - Tom Rutherford, Alan Manne
- MCP Solvers
  - MILES Tom Rutherford
  - PATH Michael Ferris, Stephen Dirkse, Todd Munson

Lifecycle Model

- A lifecycle savings investment model in which there is income uncertainty maximizing the discounted expected utility
- Utility function: Logarithm of consumption
- Version 1: Borrowing and savings
- Version 2: Only savings

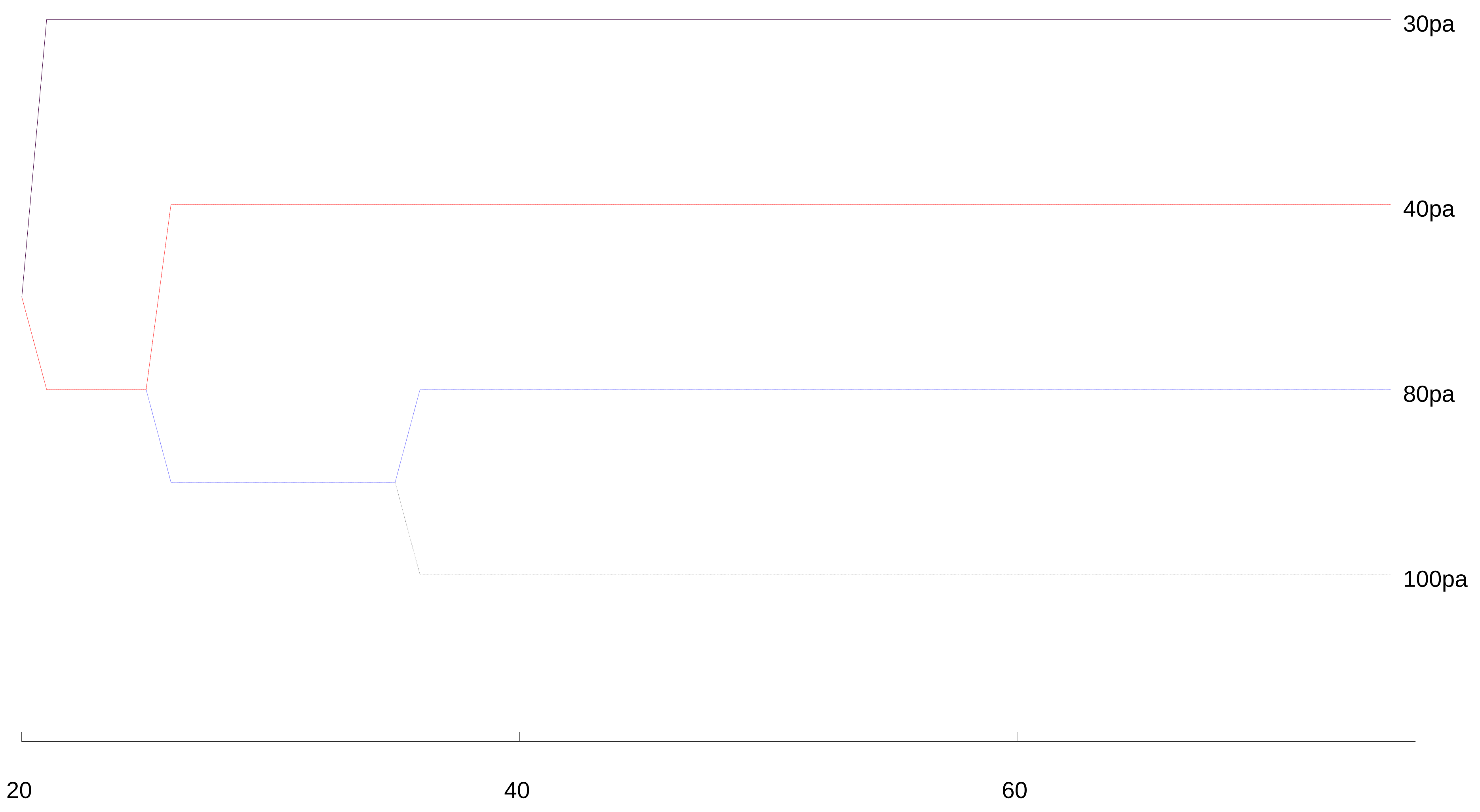
Earnings Profiles in Four States of Nature

Lifecycle Consumption-Savings Model



Tree Plot of Lifecycle Probability Tree

gams treeplot

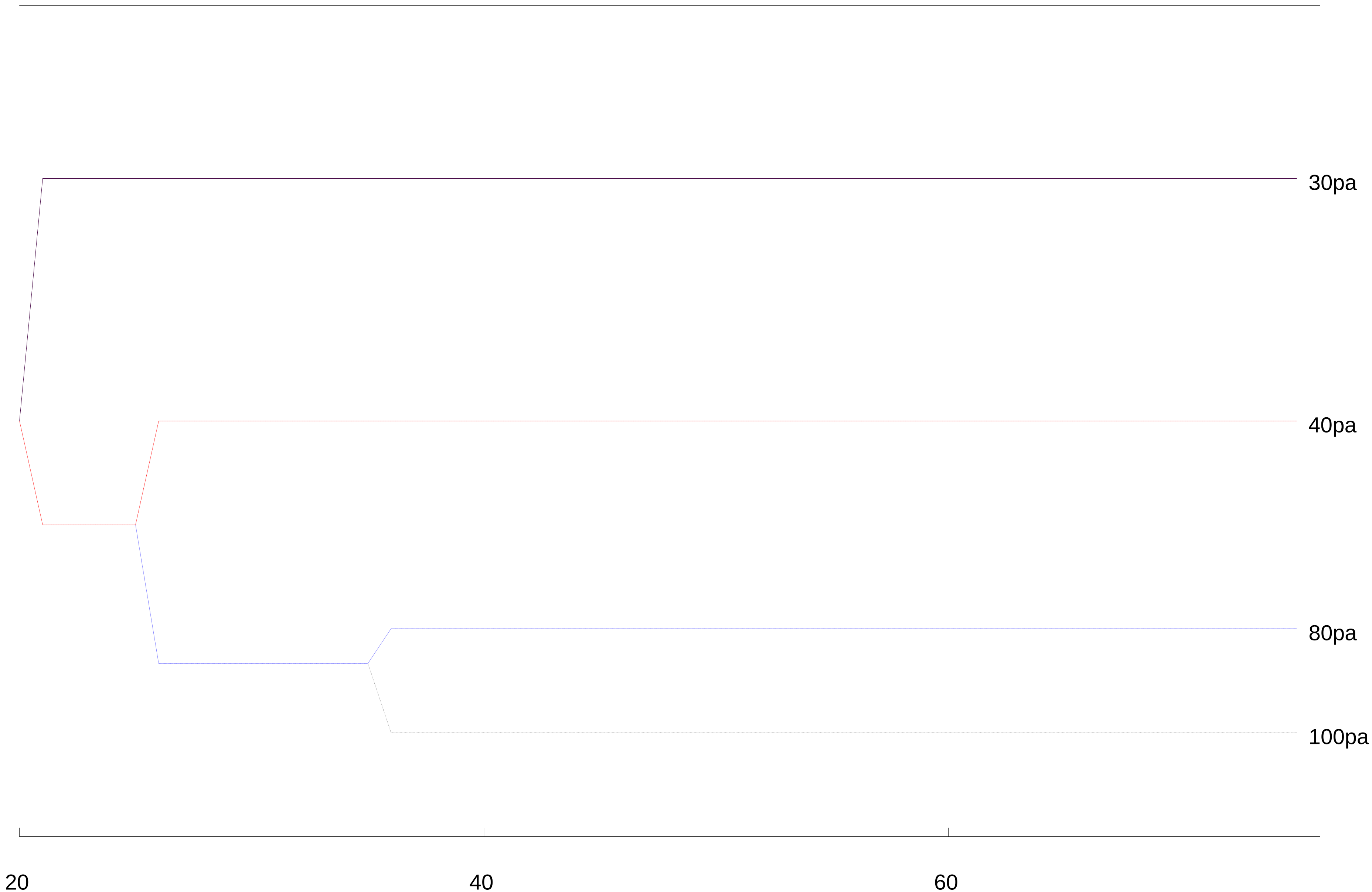


Transition in GAMS

```
sets
  t   Time periods for a typical life-cycle (ages) /20*75/

  sw  States of world / 30pa peak earn  30K per year
                        40pa peak earn  40K per year
                        80pa peak earn  80K per year
                       100pa peak earn 100K per year /

  transition(t,sw,sw)     State transitions /
   20.40pa. 30pa  Learn at age 20 if you are going to college
   25.40pa. 80pa  Learn at age 25 if you earn a PhD degree
   35.80pa.100pa  Learn at age 35 that you are good at business /

parameter pi(sw) Subjective probability /
                        30pa 0.3
                        40pa 0.4
                        80pa 0.2
                       100pa 0.1 /
```

Tree Plot of Lifecycle Probability Tree

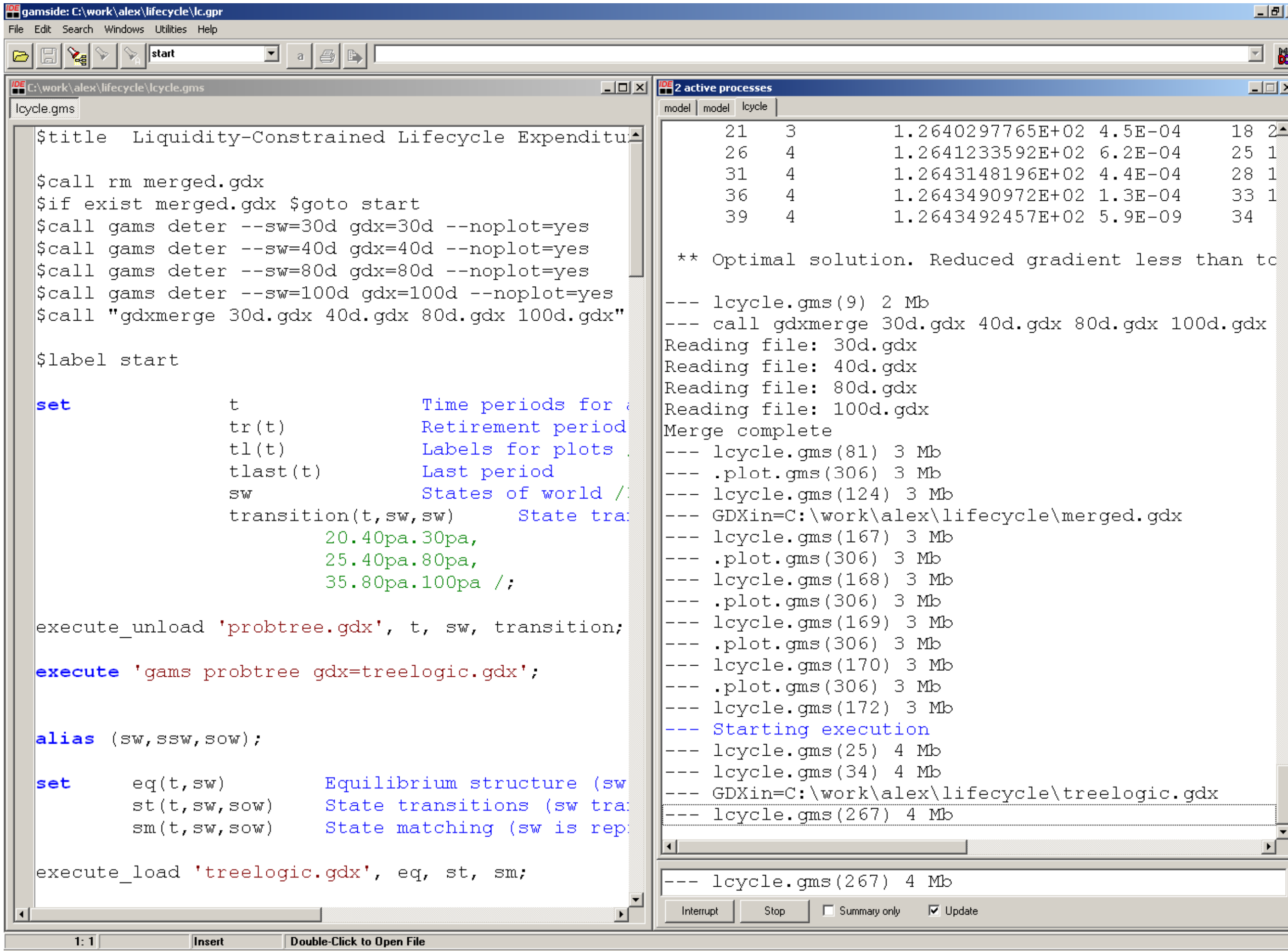
gams treeplot --piscale=yes

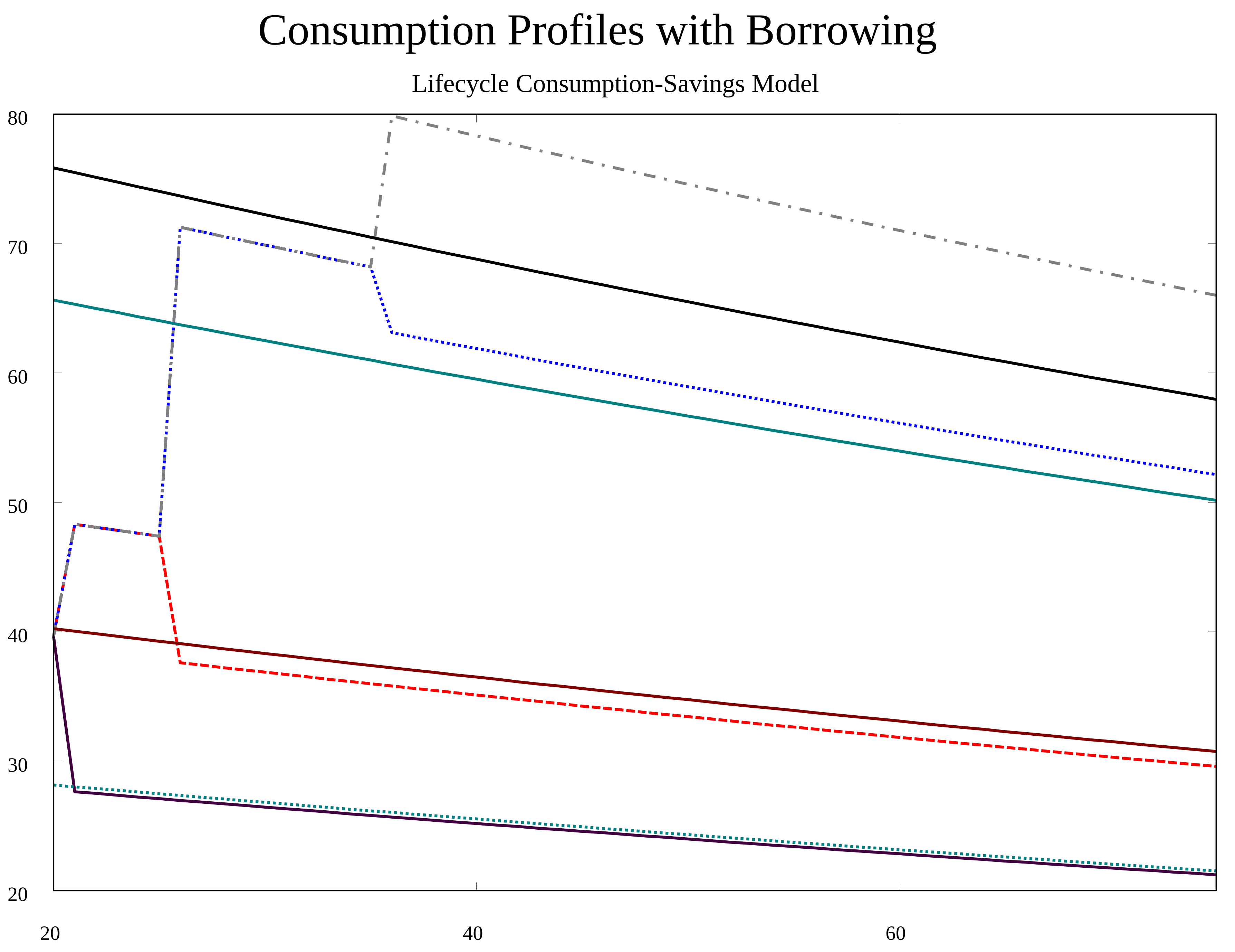


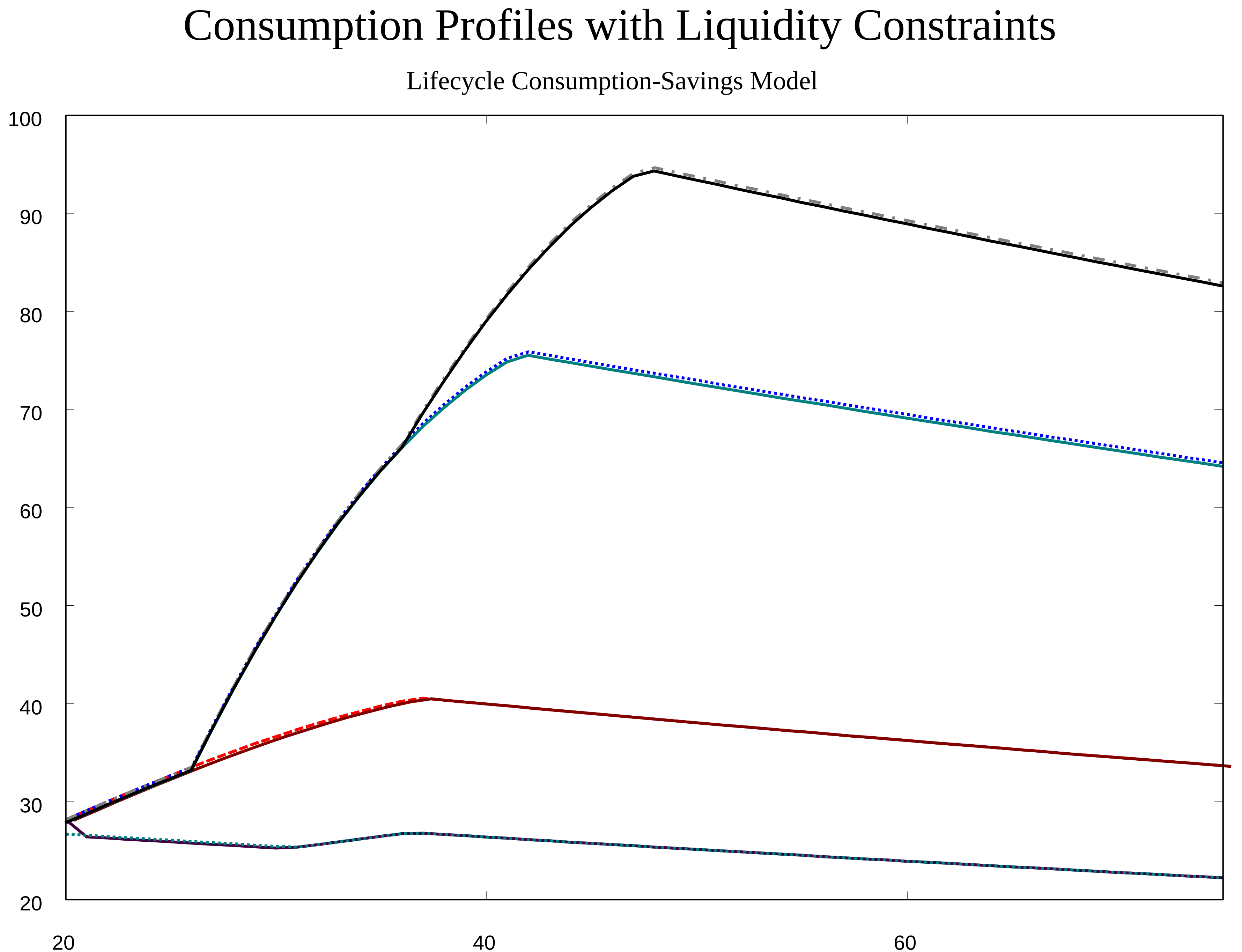
Data Structures for Stochastic Programming

```
sets
  eq(t,sw)      Equilibrium structure: sw is active in t
  st(t,sw,sow) State transitions: sw transitions to sow in t
  sm(t,sw,sow) State matching: sw is represented by sow in t
  sp(t,sw)      Preceding state of world

* Call Tom's probtree utility
$batinclude probtree t sw transition eq st sm

loop(st(t,sow,sw), sp(t,sw) = ord(sw)-ord(sow));

b.lo(t,sw)=-inf; b.up(t,sw)=+inf; // Borrowing and saving
b.fx(tlast,sw) = 0; // No debt or savings at the end

solve lcycle using nlp maximizing eu; // solve stoch. model

loop(sw, pi(sow)=0; pi(sw)=1; // Solve deterministic for sw
 solve lcycle using nlp maximizing eu );
```

Live Runs - Lifecycle Model



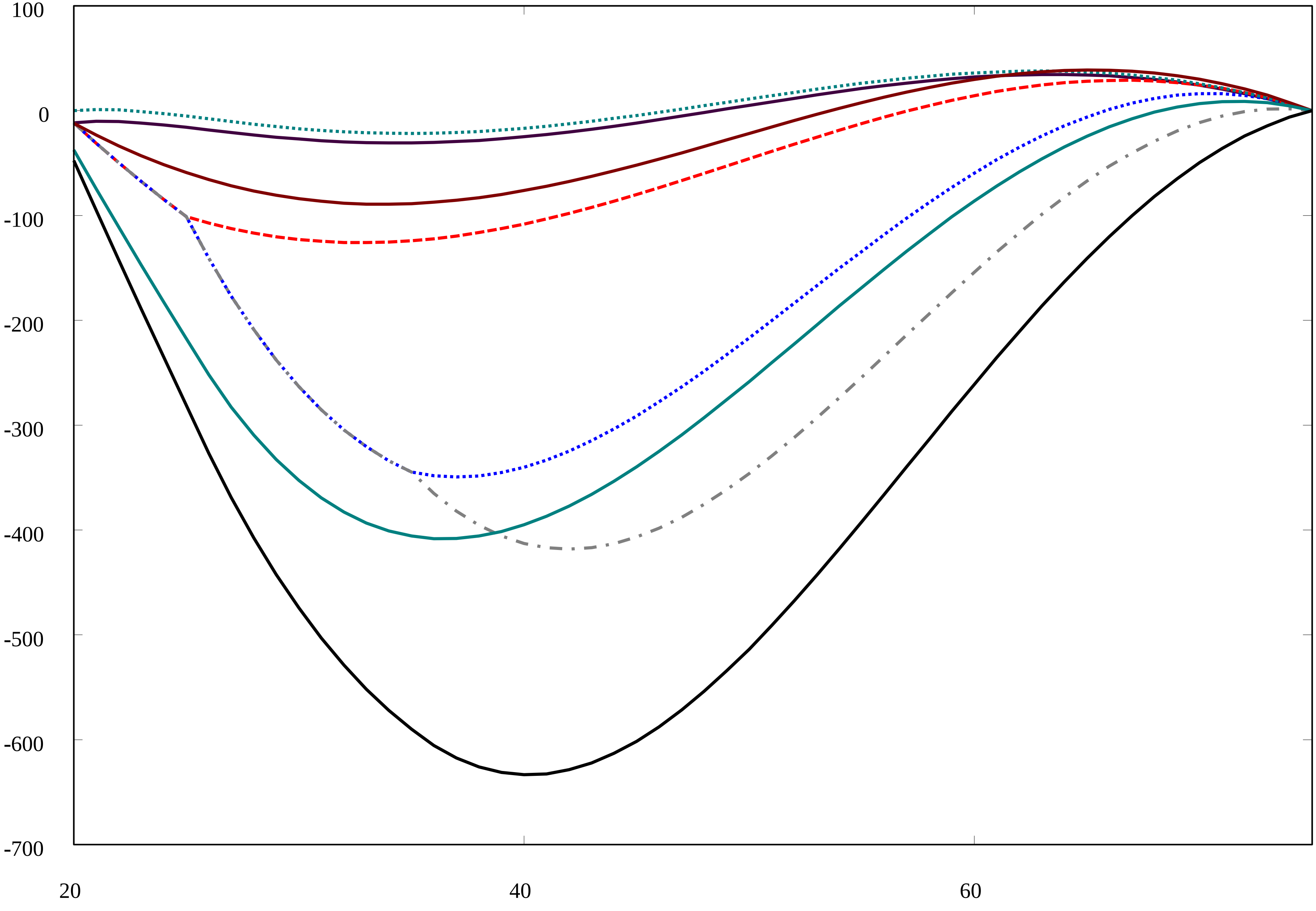

Consumption Profiles with Borrowing
Lifecycle Consumption-Savings Model
80
70
60
50
40
30
20
20
40
60



Consumption Profiles with Liquidity Constraints
Lifecycle Consumption-Savings Model
100
90
80
70
60
50
40
30
20
20
40
60

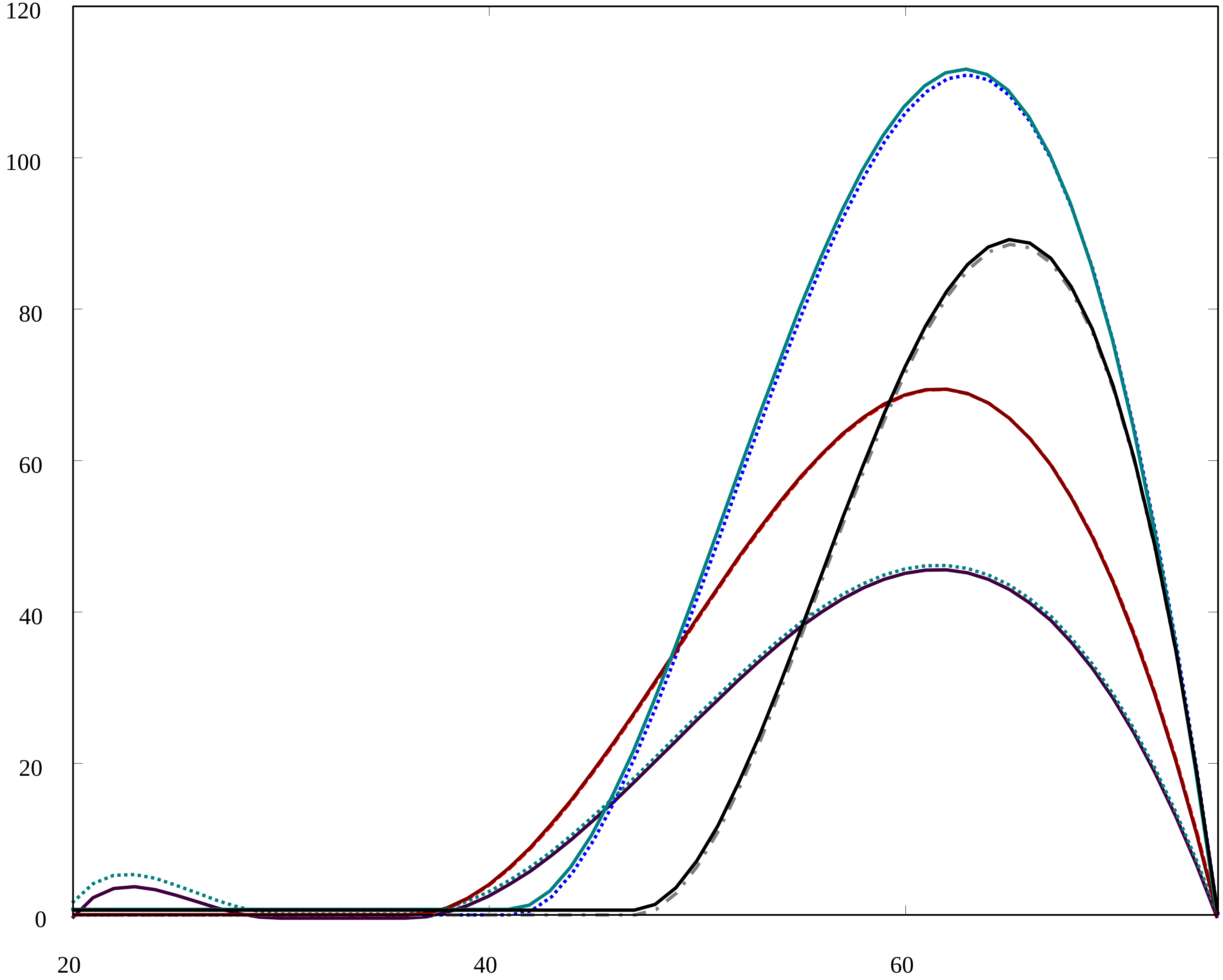
Asset Balances with Borrowing

Lifecycle Consumption-Savings Model



Asset Balances with Liquidity Constraints

Lifecycle Consumption-Savings Model



Lessons Learned

1. Event tree representation via state transitions using **`probtree`** utility
2. Event tree visualization using **`treeplot`** utility.
3. Without hedging possibilities stochasticity cannot be exploited
4. Importance of visual presentation of input and output

Classic Ramsey Model

- Ramsey Model with uncertain technology change
- Infinite horizon model in which there is an uncertain transition path to a deterministic horizon
- Motivate formulations with and without non-anticipativity constraints

- Ramsey, F P, A Mathematical Theory of Saving. Economics Journal (1928). Also see GAMS Model library *www.gams.com/modlib/libhtml/ramsey.htm*

The Ramsey Model implicit algebra

$$\max E\left(\sum_{t=0}^{\infty} \beta_t u(\tilde{c}_t)\right)$$

s.t.

$$\tilde{c}_t + \tilde{i}_t = \tilde{y}_t$$

$$\tilde{y}_t = f(\tilde{k}_t)$$

$$\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + \tilde{i}_t$$

$$\tilde{i}_t \geq 0$$

Data Structure for Stochastic Programming

s Set of scenarios, associated with all leaves at the bottom of the event tree.

SW

t Set of time periods in the model. There must be at least as many time periods in the model as there are levels in the event tree.

t

Σ_t Each node in the event tree is labelled with a scenario index s. Set Σ_t defines the set of "active scenarios" in time period t. If a scenario has appeared in time period t, it must be present in all subsequent time periods:

eq

$$s \in \Sigma_t \Rightarrow s \in \Sigma_\tau \forall \tau > t$$

Data Structure for SP - cont.

Γ_{st} **st** Defines the scenarios which branch on from scenario s in time period t. Once a scenario has "appeared", it must follow itself in all subsequent periods, i.e.:

$$s \in \Sigma_t \Rightarrow s \in \Gamma_{s\tau} \forall \tau \geq t$$

Ω_{ts} **sm** Defines the set of scenarios which branch from scenario s or from a "descendent of s" in time period t or in any later period:

$$s' \in \Omega_{st} \Rightarrow \exists t' \geq t \text{ such that } s' \in \Gamma_{st'}$$

Ramsey Model
Conventional Explicit Syntax

$$\max \sum_s \pi_s \left(\sum_{t=0}^{\infty} \beta_t u(c_{st}) \right)$$

s.t.

$$c_{st} + i_{st} = y_{st}$$

$$y_{st} = f(k_{st})$$

$$k_{s',t+1} = (1-\delta)k_{st} + i_{st} \quad \forall s' \in \Gamma_{st}$$

and *non-anticipativity constraints*:

$$\left. \begin{array}{l} c_{s'\tau} = c_{st} \\ k_{s'\tau} = k_{st} \\ y_{s'\tau} = y_{st} \\ i_{s'\tau} = i_{st} \end{array} \right\} \forall s' \in \Omega_{st}, \tau \leq t$$

Ramsey Model
Tight Formulation

$$\max \sum_s \pi_s \left(\sum_{t=0}^{\infty} \sum_{(s' | s \in \Omega_{ts'})} \beta_t u(c_{s't}) \right)$$

s.t.

$$c_{st} + i_{st} = y_{st} \qquad \forall s \in \Sigma_t$$

$$y_{st} = f(k_{st}) \qquad \forall s \in \Sigma_t$$

$$k_{s',t+1} = (1-\delta) k_{st} + i_{st} \qquad \forall s' \in \Gamma_{st}, \forall s \in \Sigma_t$$

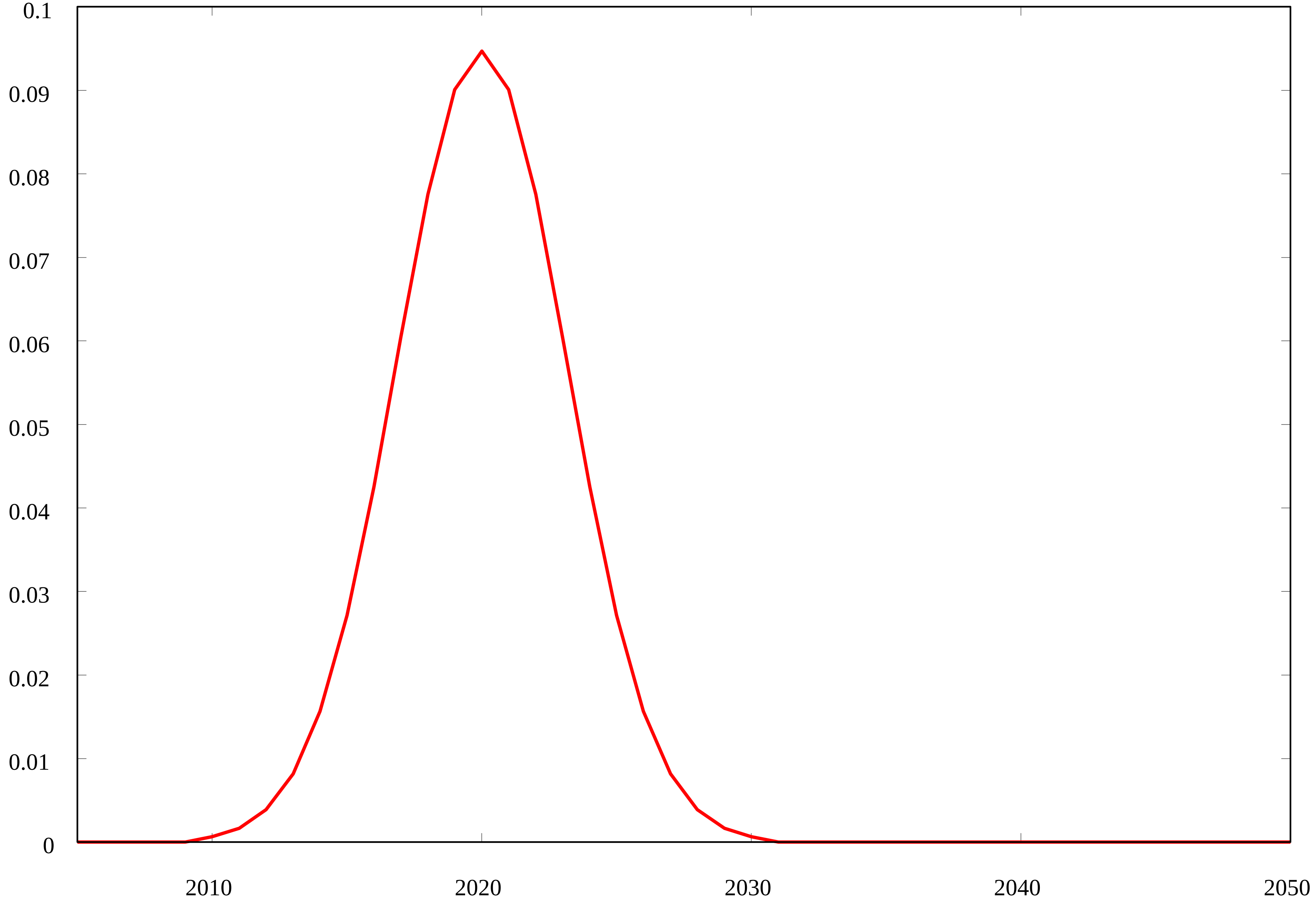
Note that the *non-anticipativity constraints* are unecessary.

Recovery of Solution from a tight Formulation

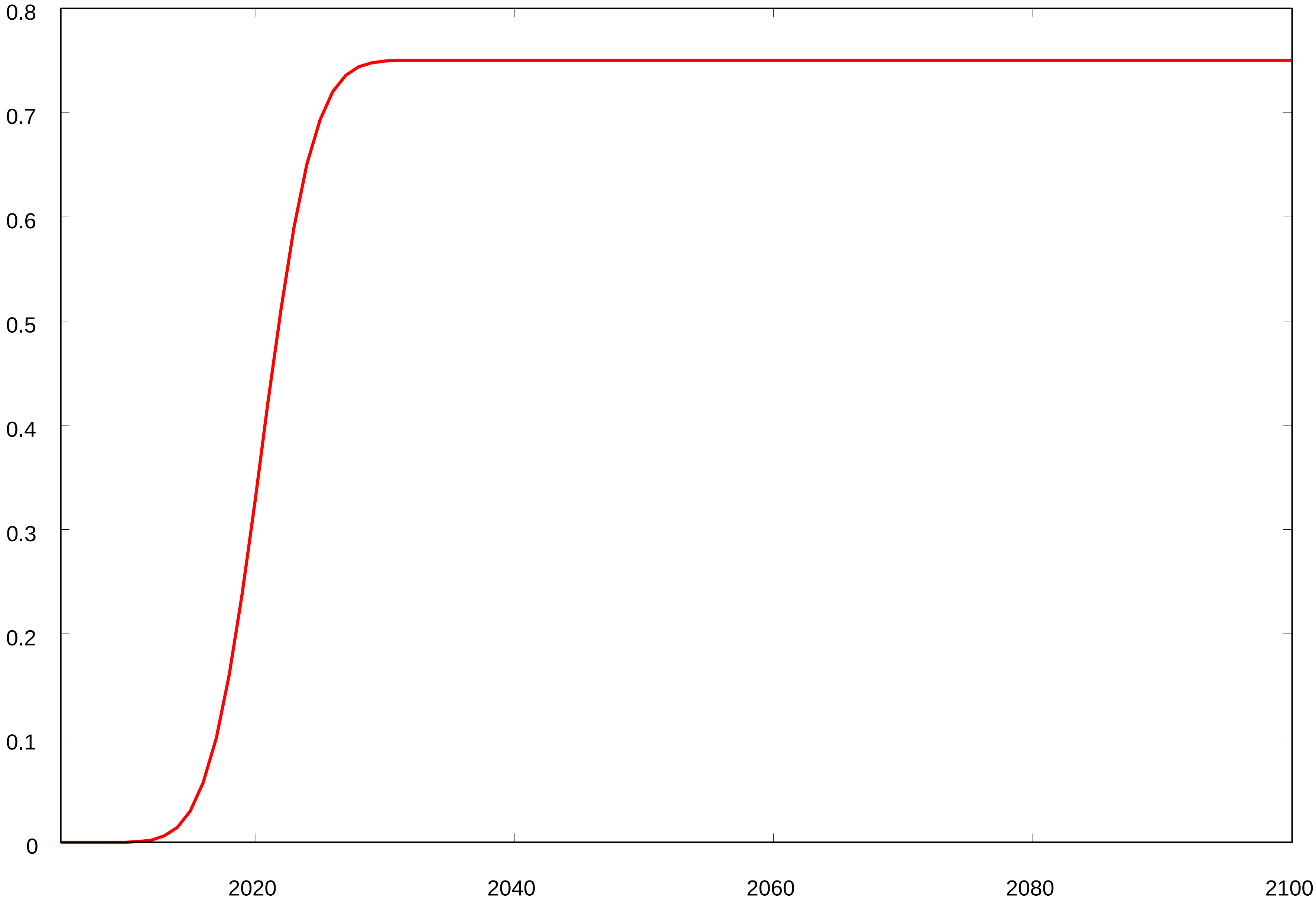
$$\left.\begin{aligned} c_{s'\tau} &\leftarrow c_{st} \\ k_{s'\tau} &\leftarrow k_{st} \\ y_{s'\tau} &\leftarrow y_{st} \\ i_{s'\tau} &\leftarrow i_{st} \end{aligned}\right\} \forall s' \in \Omega_{st}, \tau \leq t$$

Assumed Probability Distribution for Date of Technical Change

Ramsey Growth Model



Cumulative Probability Density Function



GAMS Code
Building the Event Tree

```
set t   Time periods in the model /2005*2100/
    sw  States of world /2010*2030,never/
    transition(t,sw,sw) State transitions;

* Define the event tree by specifying which states
* generate transitions. In this case 2010 defines the root
* node of the tree.

transition("2010","2010","2011") = yes;
loop((t,sw)$transition(t,sw,sw+1),
  transition(t+1,sw+1,sw+2) = yes);

pi(sw) = exp(-var*sqr(ord(sw)-card(sw)/2)); // Normal distribution

•Normalize:

pi("never") = 0; pi(sw) = 0.75*pi(sw)/sum(sow,pi(sow));
pi("never") = 1 - sum(sw,pi(sw));
```

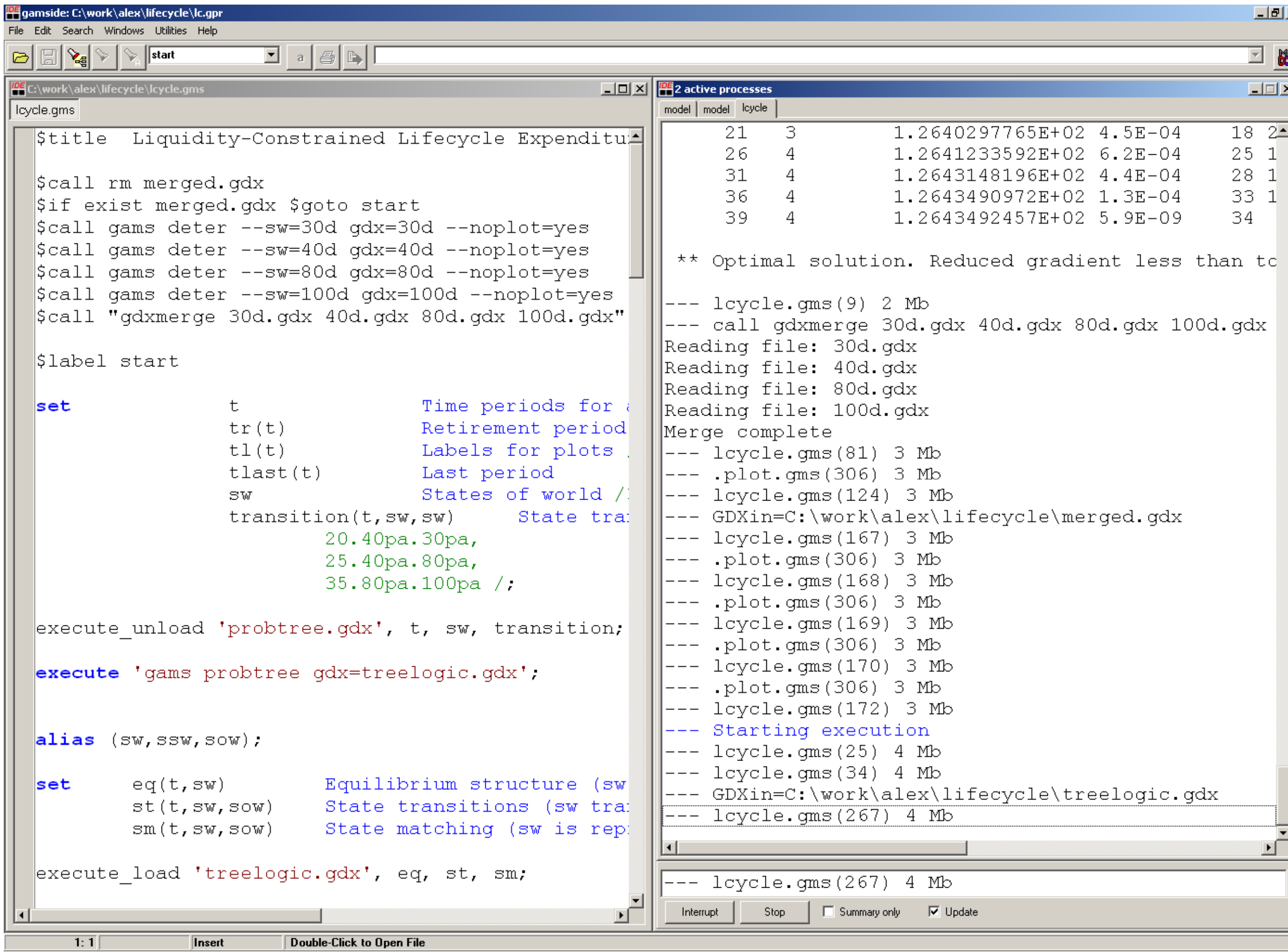
Stochastic Structure

Ramsey Growth Model

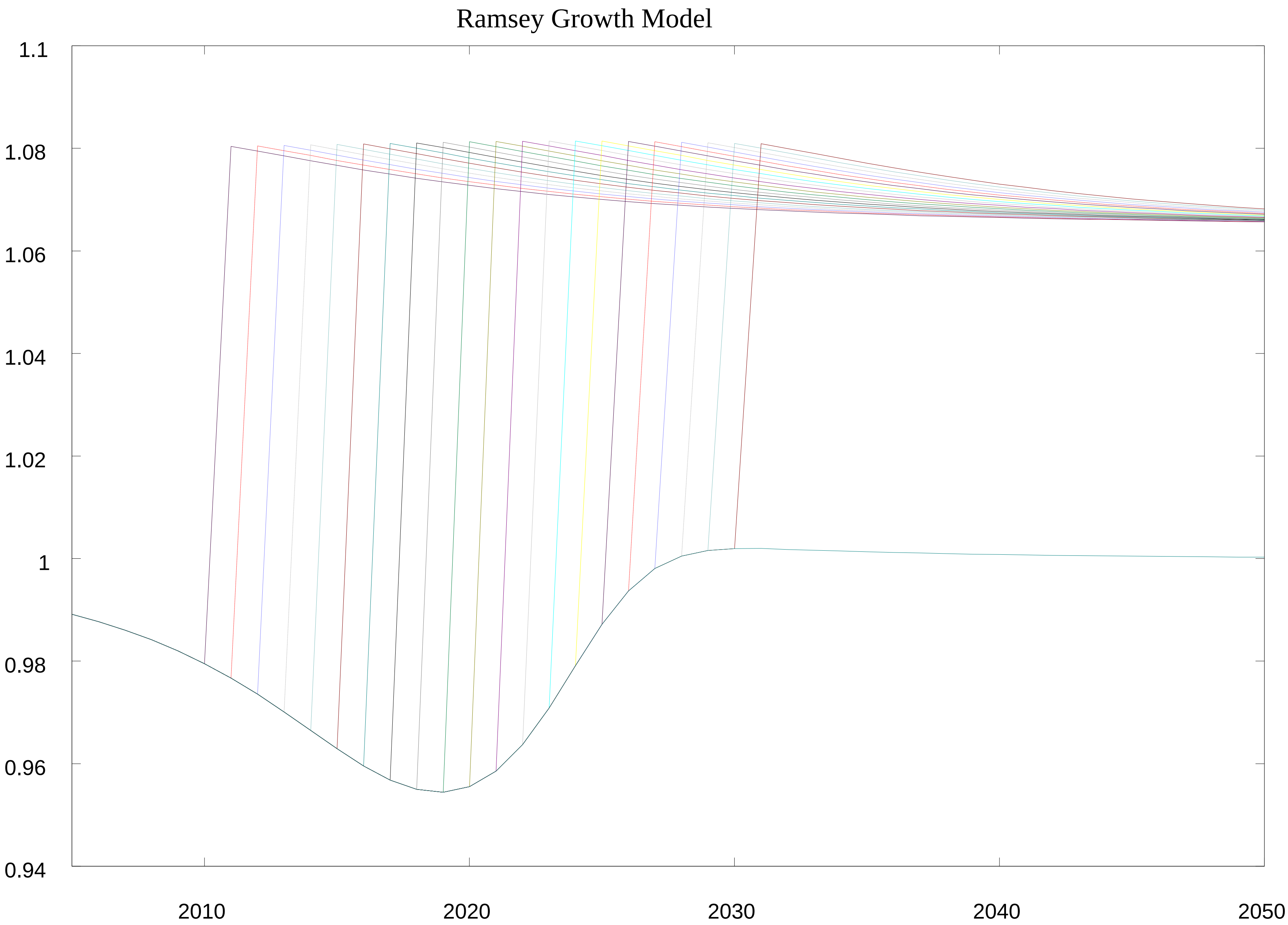
Stochastic Structure

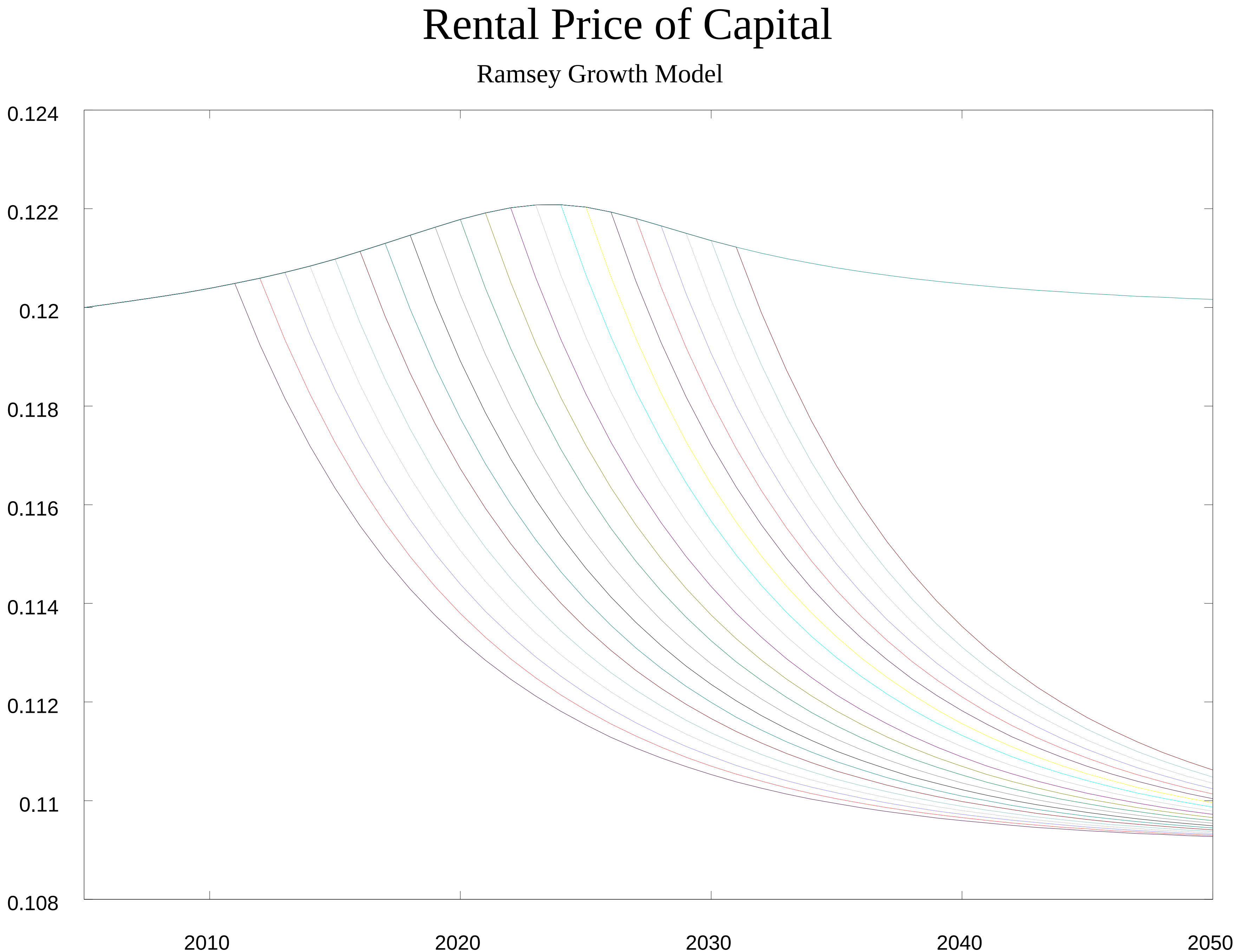
Ramsey Growth Model (--piscale=yes)

Live Runs – Ramsey Model

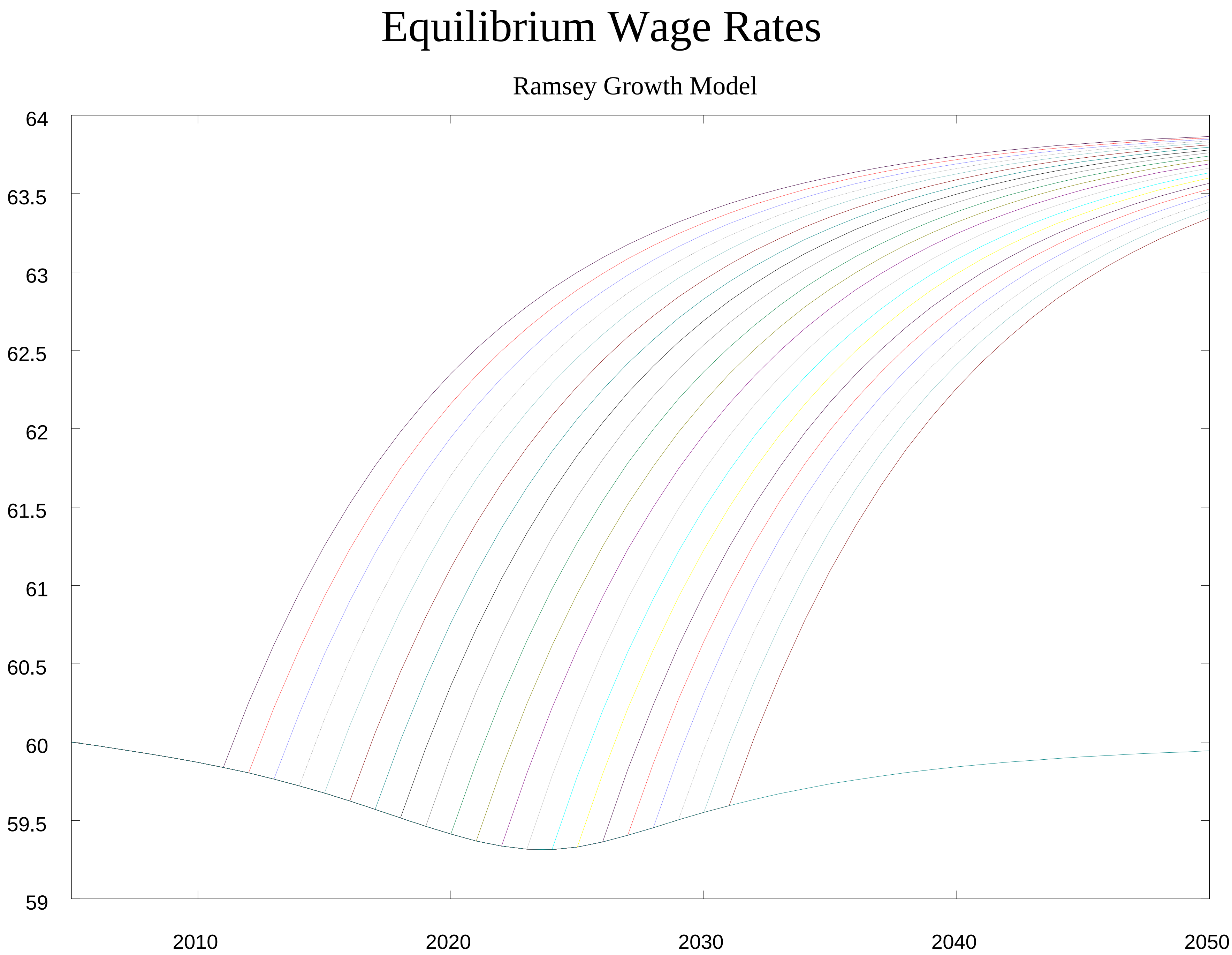


Investment


Ramsey Growth Model
1.1
1.08
1.06
1.04
1.02
1
0.98
0.96
0.94
2010
2020
2030
2040
2050



Rental Price of Capital
Ramsey Growth Model
0.124
0.122
0.12
0.118
0.116
0.114
0.112
0.11
0.108
2010
2020
2030
2040
2050



Equilibrium Wage Rates
Ramsey Growth Model
64
63.5
63
62.5
62
61.5
61
60.5
60
59.5
59
2010
2020
2030
2040
2050

Lessons Learned

1. Two mathematically equivalent formulations: with and without non-anticipativity constraints (NAC). Tight formulation is preferred (no NAC) with easy recovery of solution of the explicit problem (using Tom‘s tools)
2. Construction of transition matrix
3. Discretization of continuous distributions

Enhanced Ramsey Model

- Complementarity problem based on the Ramsey model
- Features an ad-valorem tax on capital services, hence there is no corresponding optimization problem
- In this application the policy is deterministic - a capital tax is applied five years in the future.

Enhanced Ramsey Model – cont.

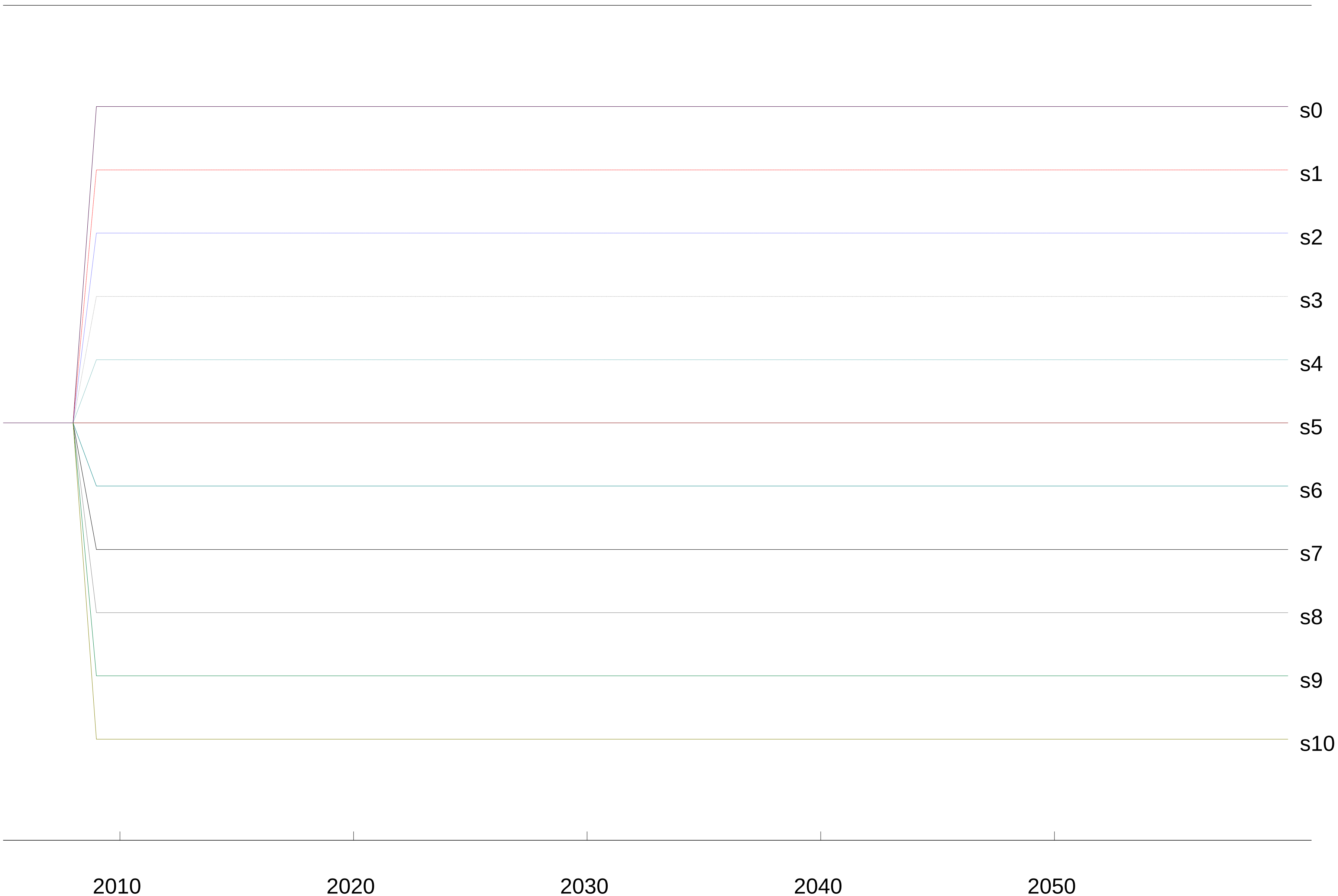
- The uncertainty concerns a model parameter - the capital-labor elasticity of substitution.
- The investors in the model only know the probability distribution of this parameter, the true value of which will only be revealed after the policy shock.
- We use the model to characterize a rational-expectations forecast of the impact of the capital tax, taking into account hedging behavior which reflects uncertainty regarding model parameters.

Stochastic Structures

```
set t      Time periods in the model /2005*2060/
    sw     States of world /s0*s10/
    transition(t,sw,sw) State transitions;

transition("2008","s0",sw)   = yes;
transition("2008","s0","s0") = no;

parameter pi(sw) Perceived probability;
pi(sw) = 1/card(sw);
```

Stochastic Structure

Capital Tax Impacts with Structural Uncertainty

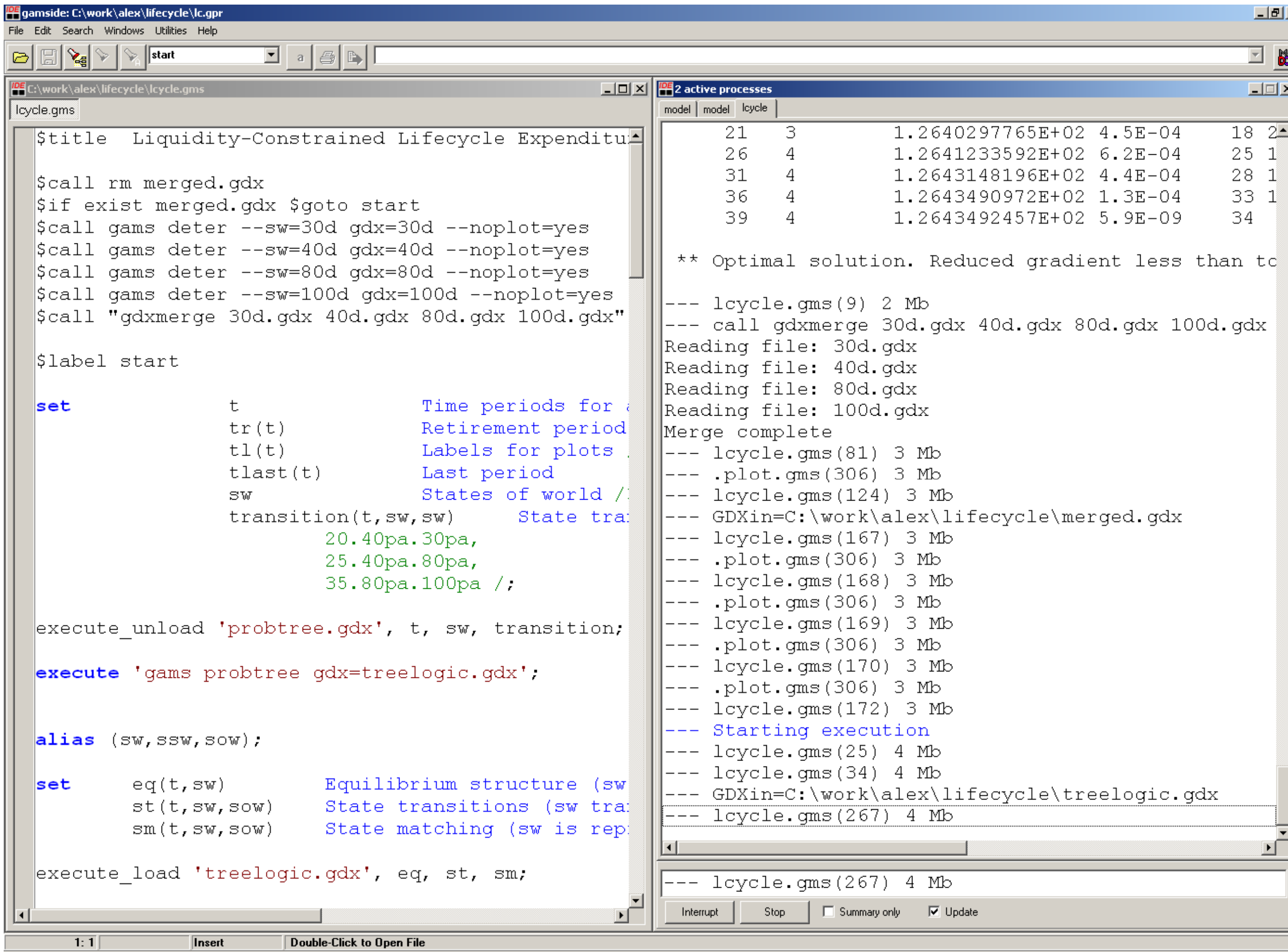


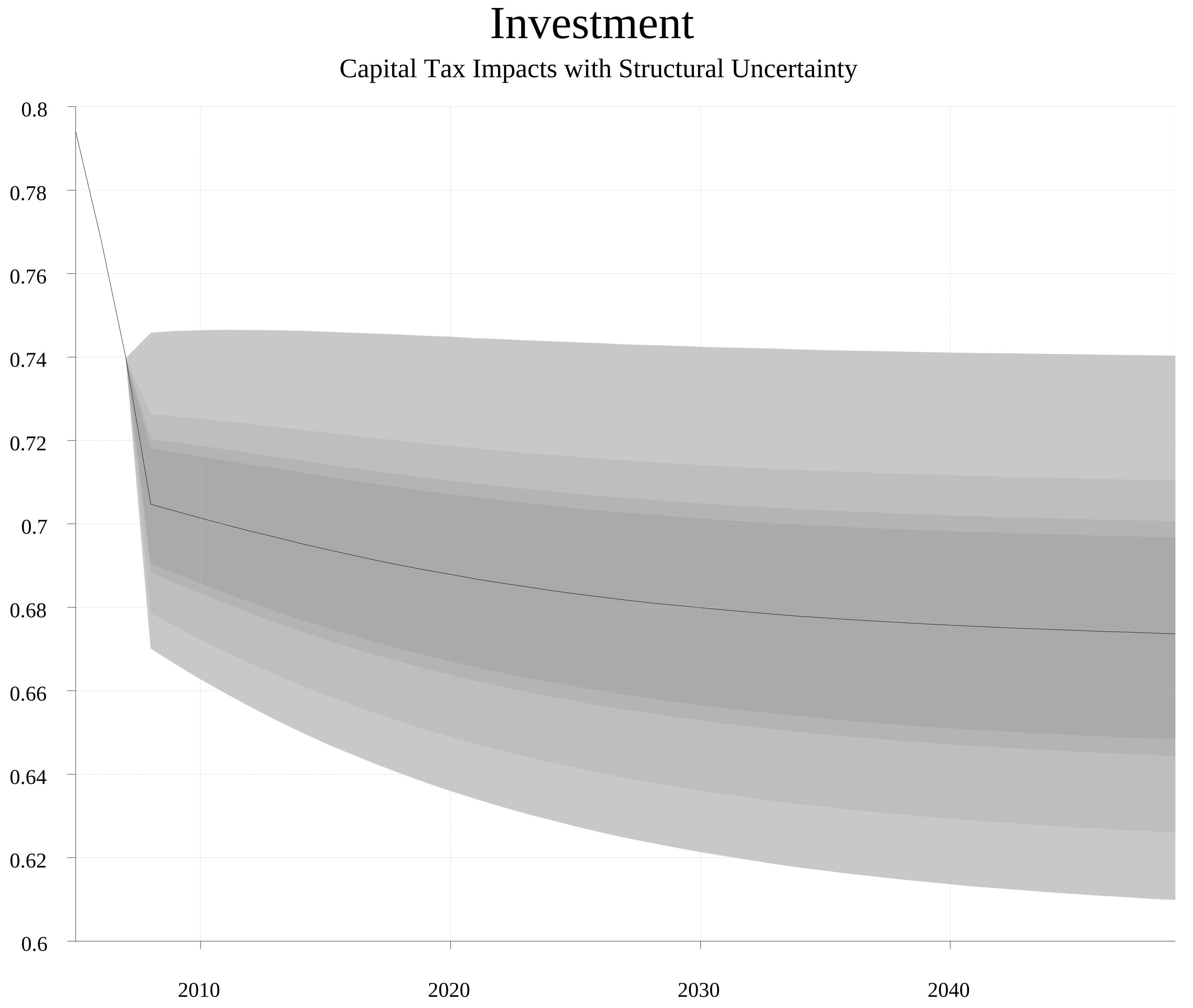
Explicit MCP Formulation

Formulation using MPSGE Sublanguage

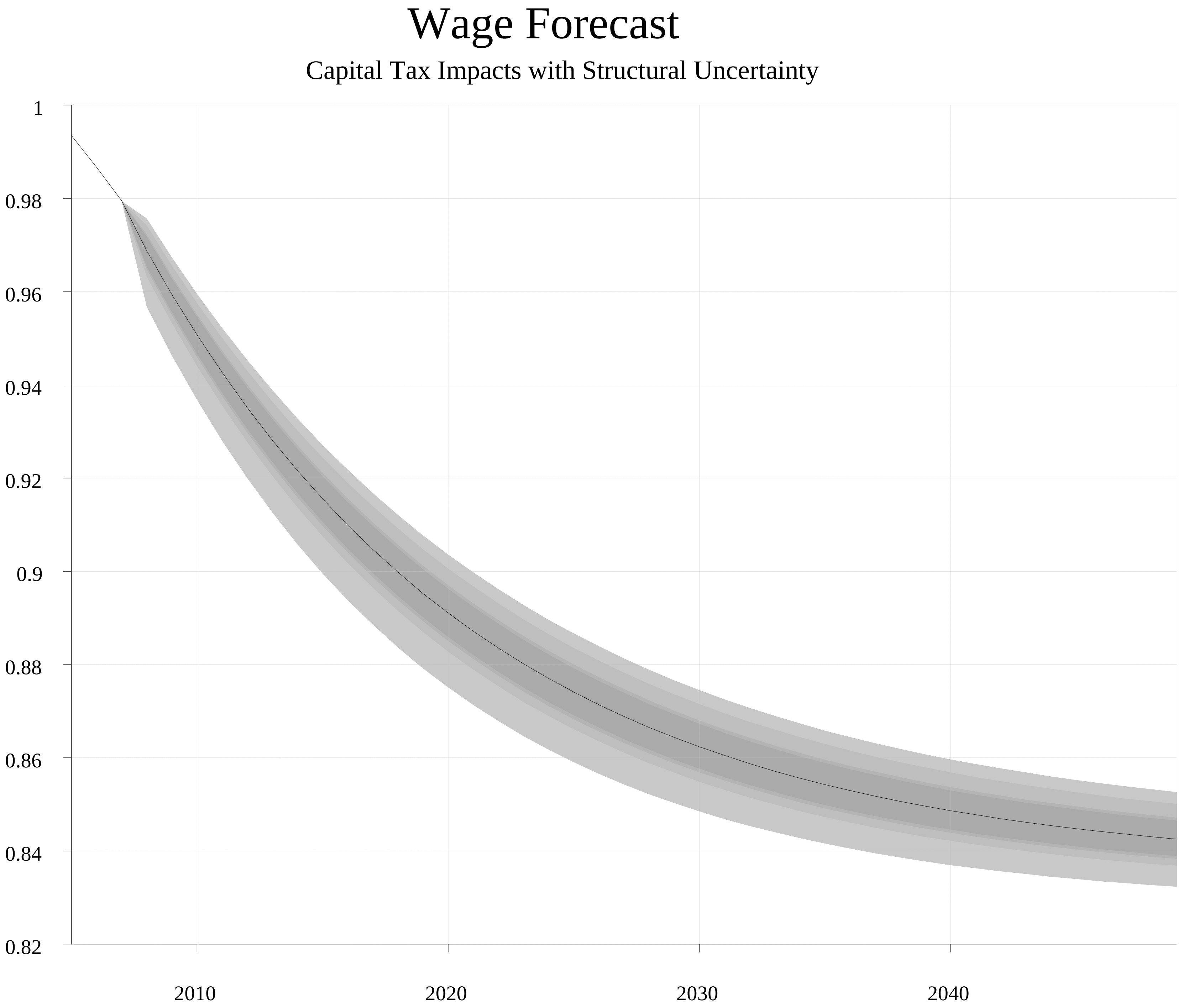
```
$model:stc

$sectors:
        u(sw)                               ! Utility
        y(t,sw)$eq(t,sw)                    ! Output
        i(tp,sw)$(t(tp) and eq(tp,sw))      ! Investment
        k(tp,sw)$(t(tp) and eq(tp,sw))      ! Capital
        c(t,sw)$eq(t,sw)                    ! Consumption
...

$prod:y(t,sw)$eq(t,sw)  s:sigma(sw)
        o:p(t,sw)       q:y0
        i:pl(t,sw)      q:l0
        i:rk(t,sw)      q:rk0   a:ra    t:tk(t)
```

Live Runs
Ramsey MCP Model



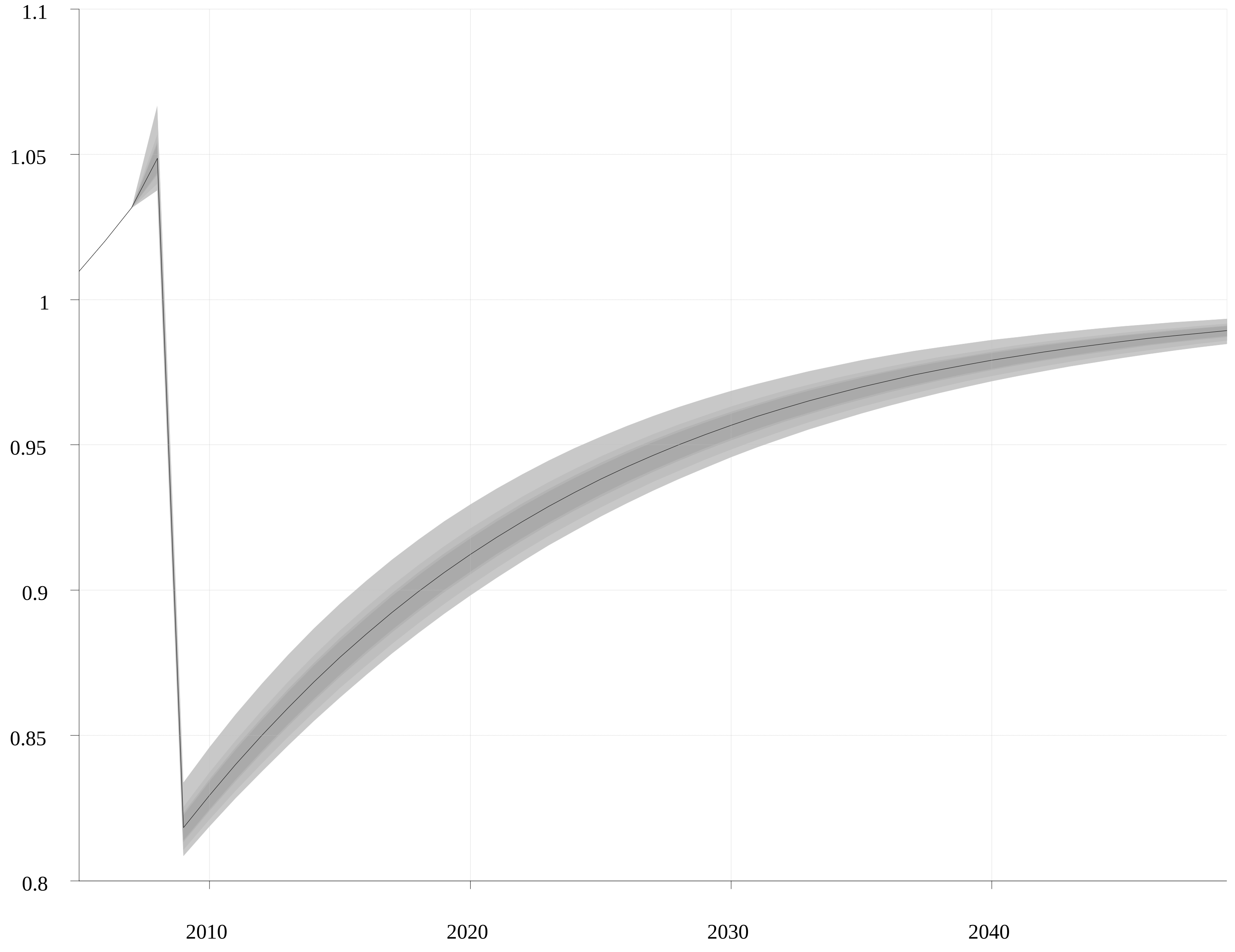


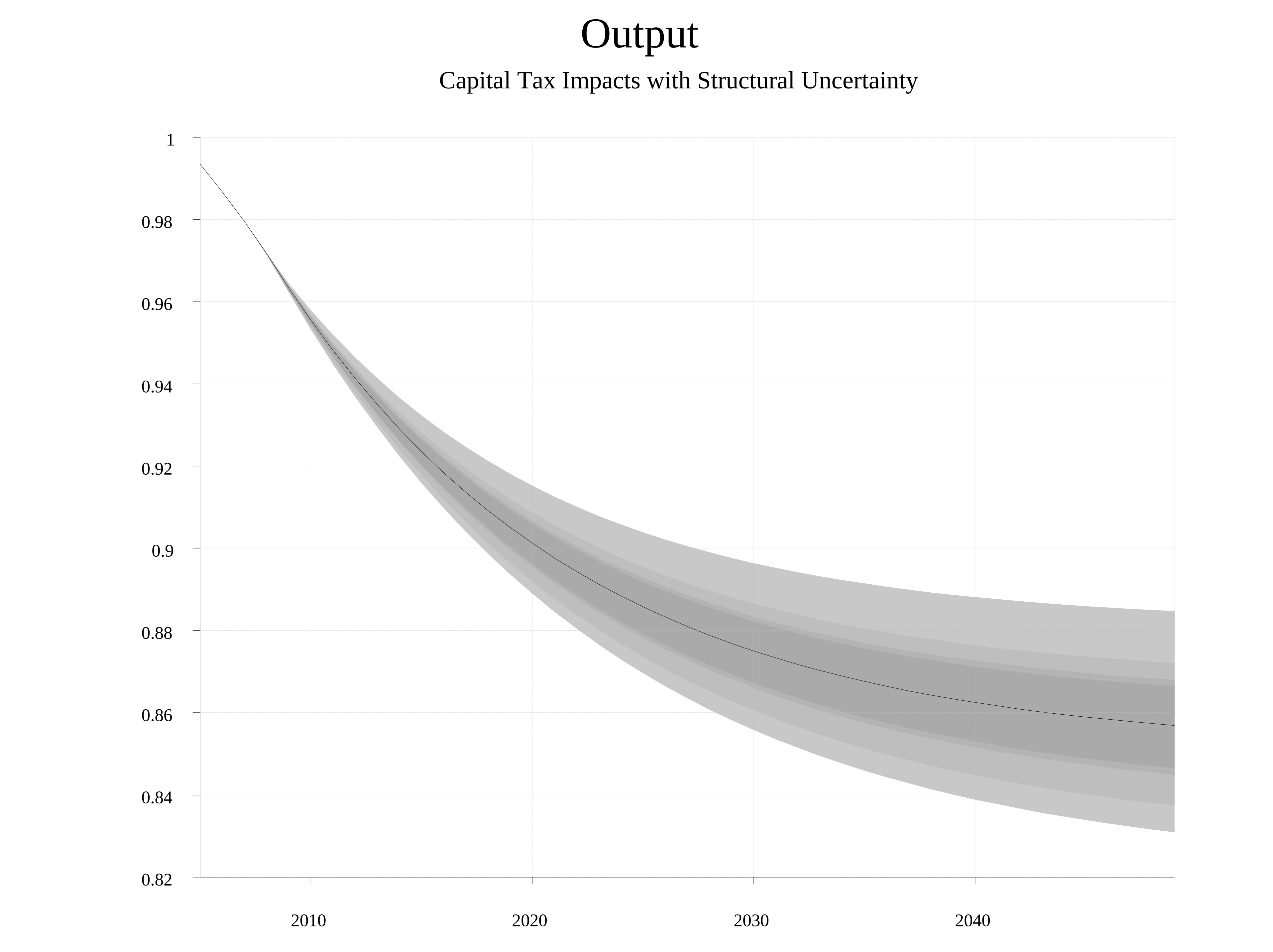
Investment
Capital Tax Impacts with Structural Uncertainty
0.8
0.78
0.76
0.74
0.72
0.7
0.68
0.66
0.64
0.62
0.6
2010
2020
2030
2040


Wage Forecast
Capital Tax Impacts with Structural Uncertainty
1
0.98
0.96
0.94
0.92
0.9
0.88
0.86
0.84
0.82
2010
2020
2030
2040

Return to Capital

Capital Tax Impacts with Structural Uncertainty




Output
Capital Tax Impacts with Structural Uncertainty
1
0.98
0.96
0.94
0.92
0.9
0.88
0.86
0.84
0.82
2010
2020
2030
2040

Lessons Learned

1. Lessons learned from NLP models carry over to MCP formulation
2. Application specific sublanguage MPSGE naturally accommodates stochastic formulations
3. Fan diagrams allow effective presentation of large number of scenarios

Summary

- Tom, please add some more comments
- The models will be available on www.mpsge.org and the GAMS web site www.gams.com
- Reproducability
- Consistent notation for optimization and complementarity