

Brief Introduction to Stochastic Programming (and Financial Modeling Applications)

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OUTLINE

- **What is SP? – “Definition”**
- **Basic Concepts – Terminology**
- **Common modeling forms/approaches of SP**
- **Need and Value of SP**
- **Overview of solution methods**
- **Sample Applications (emphasis on Finance)**
- **Active research domains**
- **Information sources**

Traditional Perceptions:

Uncertainty gives rise to Risk

Risk is “bad” – should be avoided when possible

☆Uncertainty is “bad”

BUT Uncertainties are inevitable (often exogenous)

Risk & Uncertainty should be managed,
but can not be totally eliminated

Even if risk can be eliminated, is this always a preferable strategy?
At what cost? Sacrifice of upside potential? Flexibility?

Traditional Perceptions:

“Predicting the future ... is hard (next to impossible)”.

BUT can uncertainty about the future be ignored?

Costs can be substantial.

Extreme events are bound to occur – Their effects can be catastrophic.

Themes:

Risk Management is a BIG business \$\$

Managing Risk does not mean eliminating risk.

OR/MS provide powerful tools to support improved risk management.

Key tools: Stochastic Modeling, Optimization, Derivative Pricing.

“Definition”:

Stochastic programming is a framework for modeling optimization problems that involve **uncertainty**. Whereas deterministic optimization problems are formulated with known parameters, real world problems almost invariably include some unknown parameters. When the parameters are known only within certain bounds, one approach to tackling such problems is called **robust optimization**. Here the goal is to find a solution which is feasible for all such data and optimal in some sense. Stochastic programming models are similar in style but take advantage of the fact that **probability distributions** governing the data are known or can be estimated. The goal here is to find some policy that is feasible for all (or almost all) the possible data instances and maximizes the expectation of some function of the decisions and the random variables. More generally, such models are formulated, solved analytically or numerically, and analyzed in order to provide useful information to a decision-maker.

“COSP information site”

Common Perception (“Reputation” of SP):

interesting, potentially useful, nice theory

BUT ...

conceptually difficult, impractical, computationally demanding

Undeserved today, and much more in the future:

Significant algorithmic & theoretical developments

Large-scale computational implementations

Exploitation of high-performance computing capabilities

Many practical applications – track record

Improved understanding of models

SP provides decision support tools to:

Account for uncertainty, Explore/Exploit/Reveal the value of flexibility

Overview:

- Stochastic optimization
 - Traditional
 - Small problems
 - Impractical
 - Current
 - Integrate with large-scale optimization (stochastic programming)
 - Practical examples
 - Expanding rapidly
 - Integration of financial and operation considerations

LP with random parameters (in constraints)

$$\min_{x \in \mathbb{R}_+^n} \{ cx : Ax = b, Tx \geq h \}$$

Assumptions:

- real value of (T, h) is not known:
unknown which instance of model occurs
- uncertainty is expressed by
probability distribution, e.g. "scenarios"

$$\Pr\{(T, h) = (T^s, h^s)\} = p_s, \quad s = 1, \dots, S$$

- distribution known: data, experts, ...
"deterministic LP is degenerate case"

Stochastic LP:

- Decide on x here-and-now,
without knowing the real value of (T, h) ,
but knowing its probability distribution
- $Tx \geq h$ is interpreted as a goal constraint,
to be specified more precisely

Approaches:

(1) Fat solution

Replace $Tx \geq h$ by $T^s x \geq h^s$, $s = 1, \dots, S$
Constraint to be satisfied in all scenarios

Advantage: deterministic LP model

Disadvantage:

too conservative / expensive / restrictive.
Often no feasible solution exists!

(2) Expected value

Replace $Tx \geq h$ by $\bar{T}x \geq \bar{h}$,
with $\bar{T} = \sum_s p_s T^s$ and $\bar{h} = \sum_s p_s h^s$.

Advantage: simple model again:
deterministic LP

Disadvantage: no care taken of risk:
 $T^s x \geq h^s$ only for some scenarios.

Alternative:

- use more conservative values than (\bar{T}, \bar{h})
- apply sensitivity analysis

Still poor model of decision under
uncertainty

(3) Scenario analysis

For every scenario (T^s, h^s) , $s = 1, \dots, S$, solve

$$\min_{x \in \mathbb{R}_+^n} \{ cx : Ax = b, T^s x \geq h^s \}$$

→ scenario solutions x^s

Find an overall solution by
"looking at the scenario solutions"

Advantage:

- Each scenario problem is an LP
- Improvement over EV approach ...

Very popular approach

Disadvantage: see example (later)

**BUT solutions are different for each scenario
... which one to implement?**

(4) Chance constraint

Replace $Tx \geq h$ by

$$\Pr\{Tx \geq h\} \geq \alpha$$

for some prescribed reliability level

$$\alpha \in (1/2, 1).$$

(α to be determined by problem owner)

Advantage: Risk is taken care of explicitly:

$$\text{risk} := \Pr\{Tx \not\geq h\}$$

$1 - \alpha$ is maximal acceptable risk.

Disadvantage:

discr. distr. \longrightarrow mixed-integer LP model

(In general: possibly non-convex model)

(5) Two-stage recourse model

Introduce explicitly corrective actions
(in LP framework).

Replace $Tx \geq h$ by

$$Tx + Wy \geq h$$

where y is the decision vector of a
second-stage LP problem,
value y depends on the realization of (T, h) .

(T, h)

x

$y = y^x$

Penalize corrective actions
(called recourse actions in SLP)

Minimize total expected costs

Recourse model (discrete distributions)

$$\begin{aligned} \min_{x, y^s} \quad & cx + \sum_{s=1}^S p_s qy^s \\ \text{s.t.} \quad & Ax = b \\ & T^s x + W y^s \geq h^s \quad \forall s \\ & x \geq 0, \quad y^s \geq 0 \quad \forall s \end{aligned}$$

with q unit penalty costs.

Objective: $cx +$ expected recourse costs

Advantage:

- Risk is taken care of explicitly:
risk \rightarrow expected recourse costs
- large-scale LP model

Disadvantage: too large to solve

10 independent random variables

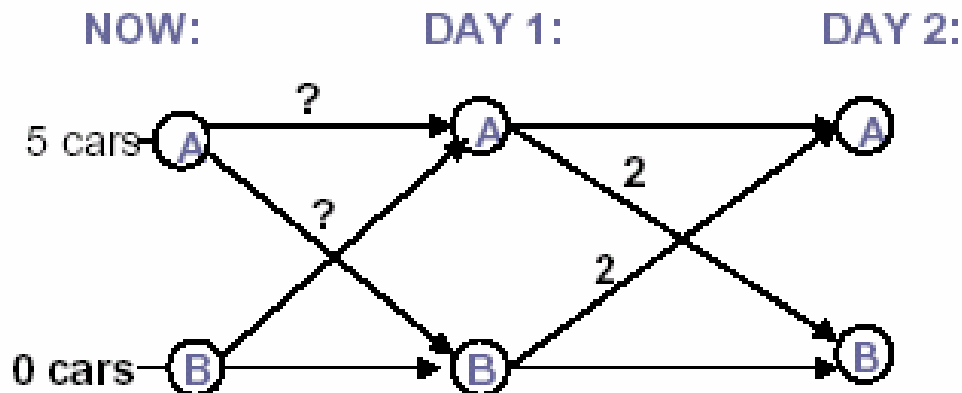
5 realizations each $\rightarrow S = 5^{10} \approx 10^7$

A is $m_1 \times n$, W is $m \times p$

\rightarrow LP with $(n + pS)$ variables,
 $(m_1 + mS)$ constraints

Vehicle allocation example

- Decision:
 - How to position empty freight cars?



DEMAND: DAY 1: B to A:Mean Value=2
DAY 1: A to B:Mean Value=2

Vehicle allocation – Mean Value Solution

Parameters: COST: 0.5 per empty car from A to B

REVENUE: 1.5 per full car from B to A, 1 from A to B

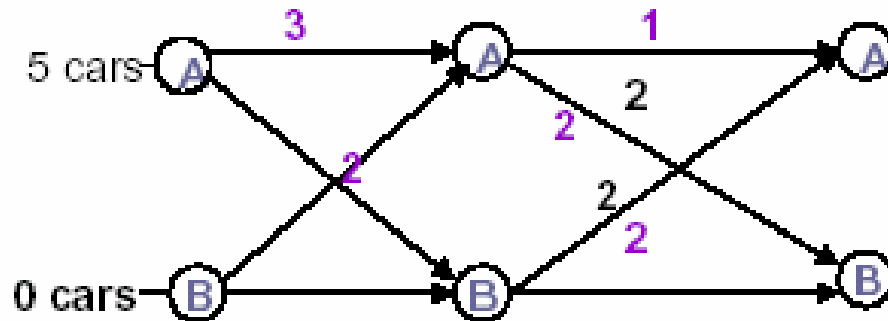
- **Maximize: Revenue-Cost**

» MOVE TWO EMPTY CARS FROM A to B

NOW:

DAY 1:

DAY 2:



RESULT: Net 2: A to B; Net 2: B to A
TOTAL(MV) = 4

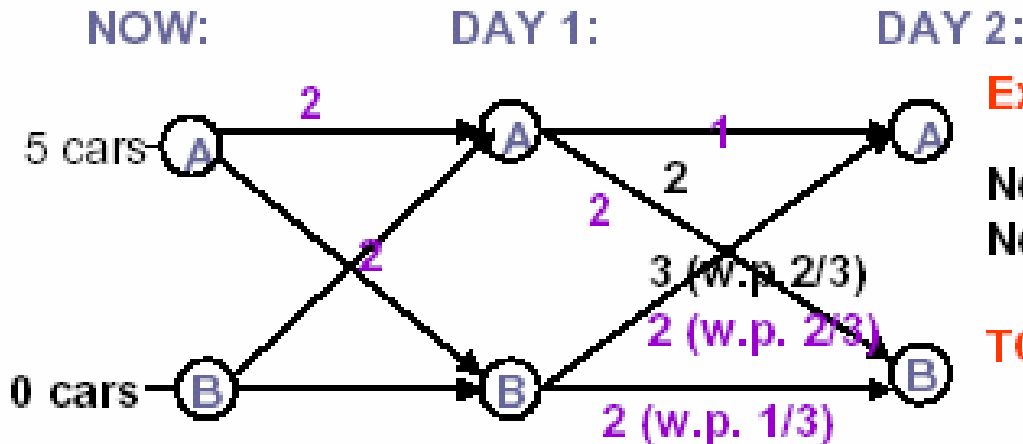
Expectation of Mean Value Model

Suppose: Demand is **Random** (Expectation from A to B=2)

- 0 from A to B with prob. 1/3
- 3 from A to B with prob. 2/3

• **Find: Expected (Revenue-Cost)**

» **MOVE Two EMPTY CARS FROM A to B**



Expected Value:

Net 2: A to B;

Net 2: B to A (w.p. 2/3)

-1: B to A (w.p. 1/3)

TOTAL (EMV): 3

Stochastic Program

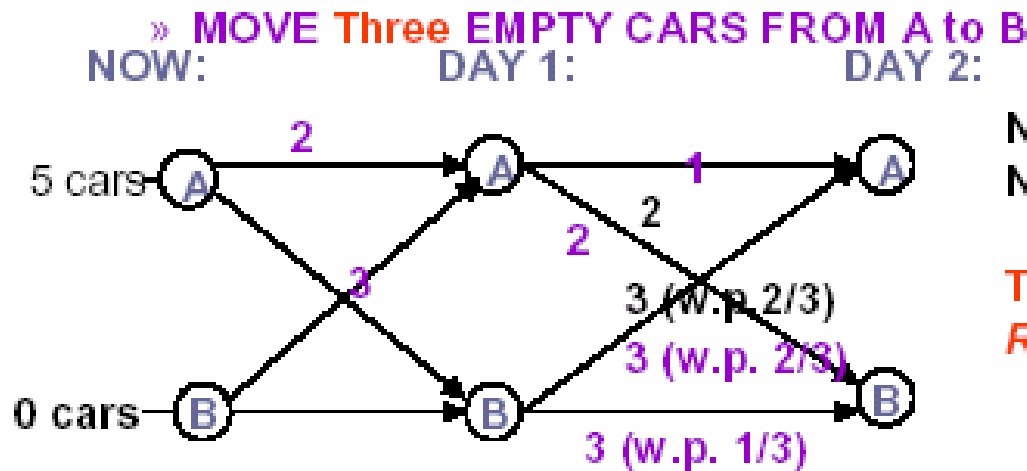
- **ASSUME:** Random demand on AB and BA
- **GOAL:** maximize **expected** profits
 - (risk neutral)
- **DECISIONS:** x_{ij} - empty from i to j
 - $y_{ij}(s)$ - full from i to j in scenario s (**RECOURSE**)
 - (prob. $p(s)$)
- **FORMULATION:**

$$\begin{aligned}
 &\text{Max } -0.5x_{AB} + \sum_{s=s1,s2} p(s) (1.5 y_{AB}(s) + 1.5 y_{BA}(s)) \\
 &\text{s.t.} \quad x_{AB} + x_{AA} = 5 \quad (\text{Initial}) \\
 &\quad -x_{AB} + y_{BA}(s) \leq 0 \quad (\text{Limit BA}) \\
 &\quad -x_{AA} + y_{AB}(s) \leq 0 \quad (\text{Limit AB}) \\
 &\quad y_{BA}(s) \leq D_{BA}(s) \quad (\text{Demand BA}) \\
 &\quad + y_{AB}(s) \leq D_{AB}(s) \quad (\text{Demand AB}) \\
 &\quad x_{AA}, x_{AB}, y_{AA}(s), y_{AB}(s) \geq 0
 \end{aligned}$$

Stochastic Programming Solution

Suppose: Demand is **Random** (as before)
 GOAL: A solution to obtain highest **expected** value

- **Maximize: Expected (Revenue-Cost)**



Expected Value:

Net 2: A to B;
 Net 3: B to A (w.p. 2/3)
 -1.5 : B to A (w.p. 1/3)
TOTAL (RP): 3.5
RP=Recourse Problem

Information and Model Value

- **INFORMATION VALUE:**
 - **FIND Expected Value with Perfect Information or Wait-and-See (WS) solution:**
 - **Know demand: if 3, send 3 from A to B; If 0, send 0 from A to B:**
 - **Earn: $2 (AtoB) + (2/3) (3) + (1/3)0 = 4 = WS$**
 - **Expected Value of Perfect Information (EVPI):**
 - **$EVPI = WS - RP = 4 - 3.5 = 0.5$**
 - **Value of knowing future demand precisely**
- **MODEL VALUE:**
 - **FIND EMV, RP**
 - **Value of the Stochastic Solution (VSS):**
 - **$VSS = RP - EMV = 3.5 - 3 = 0.5$**
 - **Value of using the correct optimization model**

Model & Information Value

- **EVPI and VSS:**
 - **ALWAYS ≥ 0 ($WS \geq RP \geq EMV$)**
 - **OFTEN DIFFERENT ($WS=RP$ but $RP > EMV$ and vice versa)**
 - **FIT CIRCUMSTANCES:**
 - **COST TO GATHER INFORMATION**
 - **COST TO BUILD MODEL AND SOLVE PROBLEM**
- **MEAN VALUE PROBLEMS:**
 - **MV IS OPTIMISTIC ($MV=4$ BUT $EMV=3$, $RP=3.5$)**
 - **ALWAYS TRUE IF CONVEX AND RANDOM**
 - **CONSTRAINT PARAMETERS**
 - **VSS LARGER FOR SKEWED DISTRIBUTIONS/COSTS**

Fundamental Concepts

Scenario Tree (Discrete representation of uncertainty)

SP Recourse Models

Non-anticipativity

Two-stage recourse model

Idea:

- allow infeasibilities w.r.t. random constraints
- correct them later, pay for corrections
- all in LP framework

ω

x

$y=y(x,\omega)$

Replace $T(\omega)x = h(\omega)$ by

$$T(\omega)x + Wy = h(\omega)$$

where $y = y(x, \omega)$ is a recourse action for

- decision x and
- realization ω

W is $m \times p$ recourse (technology) matrix

Minimize costs of recourse actions:

$y^* = y^*(x, \omega)$ is a solution (?!) of the second-stage LP problem

$$v(x, \omega) := \min_{y \in \mathbb{R}_+^p} \{qy : Wy = h(\omega) - T(\omega)x\}$$

where q is the unit recourse cost vector

$v(x, \omega)$ is the second-stage value function

Compute $v(x, \omega)$ for all $\omega \in \Omega \longrightarrow$

$$Q(x) := E_{\omega} v(x, \omega)$$

the expected recourse costs of the decision x

Minimize total expected costs $cx + Q(x)$

Recourse model (canonical form)

$$\min \{cx + Q(x) : Ax = b, x \in \mathbb{R}_+^n\}$$

$$Q(x) = E_\omega v(h(\omega) - T(\omega)x)$$

$$v(z) = \min \{qy : Wy = z, y \in \mathbb{R}_+^p\}$$

Special case: $\Omega = \{\omega^1, \dots, \omega^S\}$, prob. p_s

notation: $(T(\omega^s), h(\omega^s)) = (T^s, h^s)$

$$\begin{array}{llll} \min & cx + p_1 \cdot qy^1 + \dots + p_S \cdot qy^S & & \\ \text{s.t.} & Ax & = & b \\ & T^1x + Wy^1 & = & h^1 \\ & \vdots & \dots & \vdots \\ & T^Sx & + & Wy^S = h^S \\ & x \geq 0 & y^1 \geq 0 & y^S \geq 0 \end{array}$$

Large-scale LP, size $(m_1 + mS) \times (n + pS)$,
with special structure

Any distribution: approximate by finite one
→ solve recourse problems as large-scale LP

In general, not possible since S astronomical

"Conclusions" for recourse models:

- flexible model (LP framework)
- explicit modeling of risk (quantitative)
- specification of (g, W) may be non-trivial
- concept not easy (problem owners, cf. CC)

Nice model, but can it be solved?

THEORY:

- convex objective (see later),
linear constraints \rightarrow convex model
- nice theory (KKT), many good algorithms

Answer: YES

PRACTICE:

Function evaluations / gradients of
 $Q(x) = E_{\omega} v(x, \omega)$ very expensive.

For given x

- solve 2nd-stage LP for each $\omega \in \Omega$
- calculate E_{ω} (r -dim. integral)

Answer: NO, except in special cases

- finite distribution, small S
- 2nd-stage LP nice:
 \rightarrow special recourse structure (g, W)

Multi-stage recourse

Recall two-stage model

$$\begin{array}{ccc} & \omega & \\ & & y=y(x, \omega) \\ \hline x & & \end{array}$$

NOW: decide on x

– ω only known in distribution

... observe ω

LATER: decide on recourse action $y(x, \omega)$

– $y(x, \omega)$ is optimal solution of 2nd-stage LP

– costs $v(x, \omega) \rightarrow$ EVF $Q(x) = E_{\omega} [v(x, \omega)]$

Essential: 2 stages $\sim \omega$ revealed all at once

– more than 2 periods possible

$$\begin{array}{cccc} & \omega & & \\ \hline x^0 & x^1 & x^2 & x^3 \\ t=0 & t=1 & t=2 & t=3 \end{array}$$

(see LN 3.2.3: Multi-period prod. planning)

Either

– correct model

– approximation of MULTI-STAGE

x^0	ω^1	x^1	ω^2	x^2	...	ω^H	x^H
$t=0$		$t=1$		$t=2$...		$t=H$

Multi-stage: $H + 1 > 2$ stages
 (H is time horizon)

- decision-observation-decision...
 ... observation-decision
- $\omega = (\omega^1, \dots, \omega^H)$ revealed
 at H different moments in time
 e.g. monthly interest rate
- $\omega^t \sim F_{\omega^t|\omega^-}$, $\Omega^t \subset \mathbb{R}^{r_t}$, $t = 1, \dots, H$
- x^t depends on (x^0, \dots, x^{t-1})
 and observations $(\omega^1, \dots, \omega^t)$

At each stage

NEW INFORMATION \longrightarrow DECISION

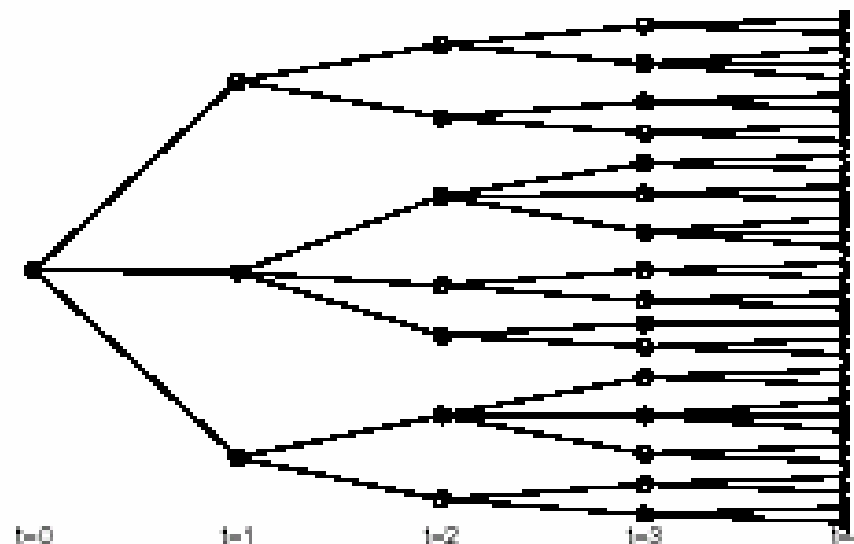
Assumptions

- discrete time t
- finite # stages
now: $t = 0$, time horizon: $t = H$
- ω^t discrete random vector, $t = 1, \dots, H$

Uncertainty represented by

$$\text{SCENARIOS } \bar{\omega} = (\bar{\omega}^1, \bar{\omega}^2, \dots, \bar{\omega}^H)$$

→ SCENARIO TREE (...)



34 scenarios (paths in tree)

In general: indexed $s = 1, \dots, S$

5 stages, decision at each node \circ

Compact formulation multi-stage recourse

$$\min_{x^0 \geq 0} \{cx^0 + Q^1(x^0) : Ax^0 = b\}$$

with, for $t = 1, \dots, H - 1$

$$Q^t(x^{t-1}) := E_{\omega^t} [v^t(x^{t-1}, \omega^t)]$$

$$v^t(x^{t-1}, \omega^t) :=$$

$$\min_{x^t \geq 0} \{q^t x^t + Q^{t+1}(x^t) : W^t x^t = h^t - T^t x^{t-1}\}$$

where $q^t := q^t(\omega^t)$, $h^t := h^t(\omega^t)$, $T^t := T^t(\omega^t)$

and $Q^H(x^{H-1}) = 0$ (...)

→ recursion

Assumptions

- complete recourse
 - sufficiently expensive recourse
 - $E_{\omega^t} [|\omega^t|] < +\infty$
- Q^t finite for all t

Solution Approaches

- **PRINCIPLES:**
 - DISCRETIZATION LEADS TO MATHEMATICAL PROGRAM BUT LARGE-SCALE
 - USE STANDARD METHODS BUT EXPLOIT STRUCTURE
- **DIRECT METHODS**
 - TAKE ADVANTAGE OF SPARSITY STRUCTURE
 - SOME EFFICIENCIES
 - USE SIMILAR SUBPROBLEM STRUCTURE
 - GREATER EFFICIENCY

Classes of SP Solution Algorithms

- Sparsity Structure Advantage
 - PARTITIONING
 - BASIS FACTORIZATION
 - INTERIOR POINT FACTORIZATION
- Similar/Small Problem Advantage
 - DP APPROACHES: DECOMPOSITION
 - BENDERS, L-SHAPED (VAN SLYKE – WETS)
 - DANTZIG-WOLFE (PRIMAL VERSION)
 - REGULARIZED (RUSZCZYNSKI)
 - VARIOUS SAMPLING SCHEMES (HIGLE/SEN Stochastic Decomposition, Abridge Nested Decomposition)
 - LAGRANGIAN METHODS

Sample Applications of SP

- **Finance (later)**
- **SCM, Logistics, Distribution, Transportation**
- **Production Planning**
- **Capacity Planning**
- **Energy (power plant management, unit commitment)**
- **Natural Resources Management**
 - Forestry, Fisheries, Agriculture, Lakes
- **Yield/Revenue Management**
- **Risk Management**

SP & Finance

Financial Modeling is a particularly fertile domain for SP.

A very active research field – significant contributions,
Continual advancements, important practical applications

Some of the Reasons:

Continual stream of challenging/important problems

Sophistication of sector – rich underlying background, receptiveness

Volatility – obvious impact of uncertainties

High availability of data

Validation potential - benchmarking

\$\$ and high stakes

Suitability of problems for SP models:

Liquid markets – recourse flexibilities

Diversity of instruments/securities (hedging instruments)

Representative SP Applications in Finance

- **Portfolio management**
Cash management, Fixed-income securities, MBS, International
- **Asset and Liability Management**
Pension funds, Insurance contracts, Personal finance
- **Financial Risk Management**
- **Design of Financial Products**
- **Real Options (Integer SP)**

Two Finalists for the Edelman Prize of Best Achievement in MS:

- 1993, Russel-Yasuda-Kasai Model (Insurance), W.T. Ziemba, et al.
- 1999, Towers-Perrin Model (Global Asset Mgmt.), J.M. Mulvey, et al.

Active Research Domains

Applications (Increasing application base and extensions)
Exploitation of Computational GRIDS
Scenario Generation Procedures (& Scenario Reduction Techniques)
Approximation Schemes & Sampling-based Methods
Modeling Tools
Bounding Methods
Output Analysis
Development of Alternative/Sophisticated Risk Measures
Financial Risk Management
Specialized Algorithms
Stochastic Integer Programming

WWW Resources on SP

- The Committee on Stochastic Programming (COSP – MPS)
“Stochastic Programming Community Homepage” <http://stoprog.org>
- Optimization Technology Center, Argonne National Lab
<http://www-fp.mcs.anl.gov/otc/Guide/OptWeb/continuous/constrained/stochastic/>
NEOS Server for Optimization: <http://www-neos.mcs.anl.gov/>
- IBM Research Group in Stochastic Programming
<http://www.research.ibm.com/stopro/>
- Stochastic Optimization Research Group,
Georgia Institute of Technology <http://www.isye.gatech.edu/so/>
- The MORE Institute, University of Arizona
<http://tucson.sie.arizona.edu/MORE/>

Groups:

- Centre for Financial Research, Judge Institute of Management, Cambridge University
<http://www-cfr.jims.cam.ac.uk/>
- Risk Management and Financial Engineering Lab, University of Florida
<http://www.ise.ufl.edu/rmfe/>
- HERMES European Center of Excellence on Computational Finance and Economics, University of Cyprus
<http://www.hermes.ucy.ac.cy>
- Centre for Analysis of Risk and Optimisation Modelling Application (CARISMA), Brunel University
<http://carisma.brunel.ac.uk/>
- Other (Un. St. Gallen, Un. Vienna, ETH-Zurich, Un. Karlsruhe, Erasmus Univ., etc.)

<http://mally.eco.rug.nl> (Maarten van der Vlerk, Univ. of Groningen,
– bibliographic database & other links)

<http://iems.northwestern.edu/~jrbirge/> (John R. Birge, Northwestern Univ.)

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P. Kall, S.W. Wallace, *Stochastic Programming*, Wiley, 1994.

Available for free: www.unizh.ch/ior/Pages/Deutsch/Mitglieder/Kall/bib/ka-wal-94.pdf

A. Prékopa, *Stochastic Programming*, Kluwer, 1995.

A Ruszczyński, A. Shapiro, *Stochastic Programming*, Series “Handbooks in Operations Research and Management Science”, Vol. 10, Elsevier, 2003.

S.W. Wallace, W.T. Ziemba (eds.), *Applications of Stochastic Programming*, SIAM, (forthcoming, 2003).

Annals of Operations Research, Kluwer:

- “Financial Modeling & Risk Management” – 2 volumes?, H. Vladimirou (ed.)
- Vol. 100 (2000), J.R. Birge, C. Edirisinghe, W.T. Ziemba (eds.)
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- Vol. 64 (1996), S.W. Wallace, J. Higle, S. Sen (eds.)

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J. Dupačová, J. Hurt, V. Stépan, *Stochastic Modeling in Economics and Finance*, Kluwer, 2002.

S.A. Zenios et al., *Practical Financial Optimization*, Blackwell, (in progress)
– 2 volumes, including GAMS models.

W.T. Ziemba, *The Stochastic Programming Approach to Asset, Liability and Wealth Management*, AIMR (forthcoming, 2003).

Handbooks in Finance, Elsevier: <http://www.elsevier.com/homepage/sae/hf/menu.htm>

- “*Heavy Tailed Distributions in Finance*”, S.T. Rachev (ed.), 2003.
- “*Asset and Liability Management*”, S.A. Zenios, W.T. Ziemba (eds.) – forthcoming (and several more).

G. Szegö (ed.), “Beyond VaR”, Feature Volume, *Journal of Banking and Finance*, 26(7), July 2002 – also a followup volume to be published by Wiley.